

Cut-off method for endogeny of recursive tree processes

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October 25, 2016

Abstract

Given a solution to a recursive distributional equation, a natural (and non-trivial) question is whether the corresponding recursive tree process is *endogenous*. That is, whether the random environment almost surely defines the tree process. We propose a new method of proving endogeny, which applies to various processes. As explicit examples, we establish endogeny of the random metrics on non-pivotal hierarchical graphs defined by multiplicative cascades and of mean-field optimization problems as the mean-field matching and travelling salesman problems in pseudo-dimension $q > 1$.¹

1 Introduction

Recursive distributional equations (RDEs) and the associated *recursive tree processes* (RTPs) occur in various problems, mainly related to branching processes, random geometry, statistical physics and combinatorial optimization, as also probabilistic analysis of algorithms. We briefly recall here some generalities, referring the reader to the excellent survey [3] for more details.

Informally speaking, recursive tree processes are analogues of Markov processes, but with each value “taking as input” several “preceding” values, forming thus a tree of dependences (whence the name). More precisely, a RTP is a rooted tree (\mathcal{T}, \bullet) of random variables $\{(X_t, \xi_t)\}_{t \in \mathcal{T}}$, where at each vertex $t \in \mathcal{T}$ the value X_t is related to the values X_{t_i} ’s at its children and to the random variable ξ_t via a given function R :

$$X_t = R(X_{t_1}, \dots, X_{t_d}; \xi_t);$$

in addition, natural independence conditions on X_t ’s and ξ_t ’s are required. We give the precise definition below, see §2.2.

Now, assume that the laws of all the ξ_t ’s are given according to the same probability measure m (this assumption will stand throughout the whole paper). Then it is natural to ask whether the laws of all the X_t ’s coincide, too. If this is the case, the RTP is called *invariant*, and the law of X_t is a solution to the RDE defined by the function R :

$$X \stackrel{d}{=} R(X_1, \dots, X_d; \xi).$$

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¹**MSC2010:** Primary 60E05, 60B10, 60E15. Secondary 81T20, 82B44, 90C27.

Key-words: recursive distributional equations, endogeny, random metrics, mean-field combinatorial optimization.

Given an invariant RTP, it is highly interesting to find out whether the values of the X_t 's are (almost surely) defined by the random environment $\{\xi_t\}_{t \in \mathcal{T}}$, that is, whether the conditional distribution of the $\{X_t\}$'s given the $\{\xi_t\}$'s is almost surely a Dirac measure. If this is the case, according to Aldous and Bandyopadhyay [3], the RTP is called *endogenous*.

Even existence of invariant RTPs (or, what is the same, existence of a solution to a RDE) is a difficult question in general. We again refer the reader to [3, § 2.2] for an overview of known methods (among which, the most widely used are monotonicity for the compact case and contraction in more general settings).

In [9], in collaboration with M. Khristoforov, we have proposed a *cut-off method* to construct *solutions* of monotone RDEs in a non-compact setting. We have applied this method in the particular case of random metrics associated with multiplicative cascades on hierarchical graphs (we recall the details below, in § 5.1). Moreover, the method is sufficiently flexible to be applied to other situations (for instance, $(\min, +)$ -type RDEs with some additional conditions imposed).

The aim of this paper is to show that the cut-off method can be modified also to prove *endogeny* of certain RTPs. As a first explicit example, we establish endogeny of the tree processes associated with random metrics defined via multiplicative cascades on non-pivotal hierarchical graphs. This has as rather reassuring consequence that the random metric spaces that were defined in [9] are measurable functions of the cascade (cf. [9, § 3.4]).

For the moment, we state our main theorem with a “black box” of assumptions, postponing the listing till § 2.4:

Theorem 1 (Endogeny via cut-off). *Suppose that the law m and the relation function R satisfy assumptions A1)–A7), listed in § 2.4, and let $\bar{\mu}$ be a solution to the corresponding RDE. Then the invariant RTP associated with $\bar{\mu}$ is endogenous.*

Its applicability for the multiplicative cascades on non-pivotal hierarchical graphs is guaranteed by the second theorem:

Theorem 2. *Assumptions A1)–A7) are satisfied for the RDEs (28) corresponding to random metrics associated with multiplicative cascades on non-pivotal hierarchical graphs with a Gaussian measure m (that is, with the log-normal random rescaling law).*

Remark 1. As in [9, § 1.5, Rem. 2], the statement of Theorem 2 holds also provided that the measure m is absolutely continuous with everywhere positive continuous density and has normal tails at $\pm\infty$.

Interestingly, different approaches by cut-off have been used to find solutions to the RDEs that are associated with problems of mean-field combinatorial optimization (see Section 6, as well as [2], [3, § 7]): the mean-field approximation for minimal matching in pseudo-dimension $0 < q < \infty$ [11, 16, 19], and for the minimum weight k -factor problem, which corresponds to the mean-field travelling salesman problem when $k = 2$ [15, 18]. For the latter, we point out that [7] makes no use of cut-off to show existence of the solution, although many ideas are in the spirit of [9], when proving that the solution is a global attractor. Relying on these works, we apply our main result to show the endogeny. For the minimal matching, the case $q = 1$ was already solved in [4]; here we use the results in [16, 19] to treat the problem of pseudo-dimension $q > 1$. This completes Aldous' program for the rigorous analysis of physical predictions for these models (see [3, § 7.5] and [8, 17]).

Theorem 3. *Assumptions A1)–A7) are satisfied for the RDE (30) corresponding to the minimum weight k -factor problem ($k \geq 1$) in pseudo-dimension $q > 1$.*

This includes the mean-field minimal matching problem ($k = 1$) and the mean-field travelling salesman problem ($k = 2$).

The structure of the paper is as follows: in Section 2 we introduce RDEs and list the assumptions for Theorem 1; in Section 3 we review the cut-off method, explaining how to adapt it to prove endogeneity. The key construction is carried out in Section 4. In Section 5 we show endogeneity for the random metric problem (Theorem 2), and in Section 6 for the mean-field optimization problems (Theorem 3).

2 Definitions and assumptions

2.1 Recursive distributional equations – Denote by \mathcal{P} the space of Radon probability measures on the compactified real line $[-\infty, \infty]$, equipped with the topology of weak convergence of measures. For the sequel, we choose and fix a metric on \mathcal{P} that defines this topology (it exists, due to the compactness of $[-\infty, +\infty]$).

Let a probability measure $m \in \mathcal{P}$ be given, as well as a function $R : [-\infty, \infty]^d \times [-\infty, \infty] \rightarrow [-\infty, \infty]$ of $d + 1$ variables; the number of variables d can be either finite or infinite. We put them in correspondence to a transformation Φ of \mathcal{P} , defined by

$$\Phi[\mu] = R_*(\mu^{\otimes d} \otimes m). \quad (1)$$

In other words, we define $\Phi[\mu]$ to be the law of the random variable

$$R(X_1, \dots, X_d; \xi),$$

where the X_i 's have law μ , ξ has law m and they are all independent.

Definition 1. A law μ satisfies the *recursive distributional equation* associated with the relation function R and the law m if it is a fixed point for the operator Φ . Alternatively, we say that the measure μ is *stationary*.

Most of the time we are interested in the case where the function R takes finite values as soon as all its variables are finite, and the measure m belongs to the subset \mathcal{P}_0 of measures supported on \mathbf{R} (that is, not charging $\pm\infty$), and after all, we want to consider stationary measures from \mathcal{P}_0 , too. However, it turns out to be quite useful to allow random variables to take infinite values during the intermediate steps of the construction (see Section 3 below).

2.2 Invariant recursive tree processes – Given a function R and a measure m as above, it is convenient to uncover the tree structure behind the RDE. We follow the presentation given in [9, § 4.1]; the reader may also consult [3, § 2.3] as a more classical reference.

Consider the rooted d -ary tree (\mathcal{T}, \bullet) ; we denote by $\|\cdot\| : \mathcal{T} \rightarrow \mathbf{N}$ the distance of a vertex to the root. Given a solution μ to the RDE associated with R and m , we define the *invariant recursive tree process* as a tree of random variables $\{(X_t, \xi_t)\}_{t \in \mathcal{T}}$ such that

- for every $t \in \mathcal{T}$ the law of ξ_t is m ,
- for every $t \in \mathcal{T}$ the law of X_t is μ ,
- for every $t \in \mathcal{T}$ the equality

$$X_t = R(X_{t_1}, \dots, X_{t_d}; \xi_t) \quad (2)$$

holds, where the t_i 's are the d children of t in \mathcal{T} ,

- for every $n \in \mathbf{N}$ the random variables $\{\xi_t\}_{\|t\| < n}$ and $\{X_t\}_{\|t\| = n}$ are independent altogether (in particular the random variables $\{\xi_t\}_{t \in \mathcal{T}}$ are all independent).

Reversely, the existence of a tree of random variables $\{(X_t, \xi_t)\}_{t \in \mathcal{T}}$ satisfying the properties above implies that the measure μ is stationary (this is an easy consequence of equality (2) and independence).

2.3 The cut-off operators – For a fixed RDE associated with a relation function R and a measure m , we introduce a family of operators Φ_a , $a \in \mathbf{R}$, as follows: considering the new relation function

$$R_a(X_1, \dots, X_d; \xi) := \min(R(X_1, \dots, X_d; \xi), a), \quad (3)$$

the *cut-off operator* Φ_a is defined as in (1). That is, the operator Φ_a sends any measure $\mu \in \mathcal{P}$ to the law of $R_a(X_1, \dots, X_d; \xi)$, where all the X_i 's have law μ , ξ has law m , and they are all independent.

2.4 Assumptions and some remarks – In this paragraph we list the assumptions for Theorem 1. Most likely, these may not be the most general assumptions that make our strategy work, but they are general enough to treat the particular problems of Theorems 2 and 3.

A1). The function R is monotone non-decreasing in each of the first d variables X_1, \dots, X_d .

Remark 2. The monotonicity of R implies that Φ preserves the *stochastic order* \preceq (recall that $\mu \preceq \mu'$ if there is a coupling (X, X') between these measures such that $X \leq X'$ almost surely).

A2). The function R commutes with the translations: for any $c \in \mathbf{R}$ and any $x_1, \dots, x_d, y \in \mathbf{R}$

$$R(x_1 + c, \dots, x_d + c; y) = R(x_1, \dots, x_d; y) + c.$$

Remark 3. Some natural processes (for instance, the one for random metrics) are rather scale-equivariant, that is, they satisfy the relation

$$R(\lambda z_1, \dots, \lambda z_d; y) = \lambda R(z_1, \dots, z_d; y).$$

For these processes, we pass to the logarithmic scale, that is, we rather consider $X_t := \log Z_t$.

A3). There exists a subset \mathcal{Q} of \mathcal{P} and a metric $d_{\mathcal{Q}}$ defined on \mathcal{Q} which induces a topology (non-strictly) finer than the weak-* topology. The push-forward of the action of \mathbf{R} by translations on $[-\infty, +\infty]$ to the space of measures \mathcal{P} preserves \mathcal{Q} and restricts to a continuous action on $(\mathcal{Q}, d_{\mathcal{Q}})$.

We also require that \mathcal{Q} contains the Dirac measures $\{\text{Dirac}_a\}_{a \in \mathbf{R}}$, and that the operators Φ and Φ_a 's, $a \in \mathbf{R}$, preserve the space \mathcal{Q} , and are continuous on this space (with respect to the topology defined by the distance $d_{\mathcal{Q}}$).

From now on, we presume that such a subset \mathcal{Q} and the distance $d_{\mathcal{Q}}$ are chosen and fixed.

A4). There exists a stationary measure $\bar{\mu} \in \mathcal{P}_0 \cap \mathcal{Q}$.

Again, consider $\bar{\mu}$ to be chosen and fixed. For any $c \in \mathbf{R}$ we set $\bar{\mu}_c$ to be the translation of $\bar{\mu}$ by c : if X has law $\bar{\mu}$, then $X + c$ has law $\bar{\mu}_c$.

Remark 4. The assumption A2) of translation-equivariance implies that all the translates $\bar{\mu}_c$ of the measure $\bar{\mu}$ are also stationary. In particular, this implies that even if such a random tree process is endogenous, in order to define the X_t 's one has to “fix a scale” by choosing one of the translates of $\bar{\mu}$: replacing it with $\bar{\mu}_c$ will add c to all the X_t 's.

A5). The space \mathcal{Q} is contained in the basin of attraction of the set of translates of $\bar{\mu}$. More precisely, for any $\mu \in \mathcal{Q}$ there exists a *center* $c = \tilde{c}(\mu) \in \mathbf{R}$ such that

$$\Phi^n[\mu] \xrightarrow{d_{\mathcal{Q}}} \bar{\mu}_{\tilde{c}(\mu)} \quad \text{as } n \rightarrow \infty. \quad (4)$$

A6). There exists a compact set $\mathcal{Q}_0 \subset \mathcal{Q}$, having $\bar{\mu}$ as an interior point, such that the *center function* $\tilde{c} : \mathcal{Q} \rightarrow \mathbf{R}$ is defined (by (4)) and continuous on \mathcal{Q}_0 , and that the convergence in (4) is uniform on \mathcal{Q}_0 .

Remark 5. The equivariance assumption A2) implies that we have an analogous convergence to the translates of $\bar{\mu}$ in a neighbourhood of any Dirac measure Dirac_a , $a \in \mathbf{R}$, and that we have $\tilde{c}(\text{Dirac}_a) \rightarrow +\infty$ as $a \rightarrow +\infty$. Also, the function \tilde{c} is in fact continuous of \mathcal{Q} , as any initial measure μ after a finite number of (continuous) iterations of Φ falls in a $\tilde{c}(\mu)$ -translate of \mathcal{Q}_0 , where the continuity of the center function is guaranteed by A6). For the same reason, the convergence (4) is uniform on compact subsets of \mathcal{Q} , wherever it takes place pointwise (in particular, on compact subsets of \mathcal{Q}).

Remark 6. In fact, as the reader will see all along the proofs, the assumption A5) can be dropped: the assumption A6) suffices. Nevertheless, we prefer to impose it for the clarity of the exposition (otherwise one has to introduce further notations and to keep track of the domains of definition, etc.).

The continuity of the center function \tilde{c} implies that a cut-off of the stationary measure $\bar{\mu}$ for a large value a has quite a small effect on the center of the measure. Our last assumption quantifies this effect:

A7). Define the function $\Delta : \mathbf{R} \rightarrow \mathbf{R}$ by setting $\Delta(a) := -\tilde{c}(\Phi_a[\bar{\mu}])$. Then Δ tends to zero superexponentially as $a \rightarrow +\infty$, that is, for any $\beta > 0$

$$\Delta(a) = o\left(e^{-\beta a}\right).$$

Remark 7. The asymmetry in the assumption A7) is due to the fact that we are using the *upper* cut-off; some functions R may require the *lower* cut-off and thus the corresponding reversion of assumptions and arguments.

3 The cut-off method revised

3.1 Earlier appearances – We begin this part by recalling some generalities about the cut-off method. It was introduced in [9, §4.3] in order to find solutions for some translation-equivariant functions R (corresponding to random metrics on hierarchical graphs defined by multiplicative cascades). A main obstacle was the drift: in fact, one had to consider a family of RDEs

$$X \stackrel{d}{=} R(X_1, \dots, X_d; \xi) + s, \tag{5}$$

parametrized by the (also unknown!) *drift constant* s . Among these parameters, there is at most one $s = s_{cr}$ for which (5) possesses a “physically reasonable” solution: for $s < s_{cr}$ the iterations of the associated Φ make every initial measure converge to $-\infty$, whereas for $s > s_{cr}$, the convergence is to $+\infty$.

To avoid this problem, the relation function R is replaced by the cut-off function R_a defined by (3), where a cut-off is applied, and consequentially Φ is replaced by Φ_a . Then the transformation Φ_a sends any measure to a measure supported on $[-\infty, a]$, and one can find a stationary measure for the new process as the (stochastically monotone) limit of iterations $\Phi_a^n[\text{Dirac}_{+\infty}]$ (cf. [3, Lemma 4]). This limit is non-trivial (that is, not concentrated at $-\infty$) if and only if $s > s_{cr}$. One then finds a stationary measure for the initial process (so $s = s_{cr}$) by taking a non-trivial “diagonal” limit as $s \searrow s_{cr}$ and $a \rightarrow +\infty$ (the latter ensures that the effect of the cut-off disappears).

3.2 Cut-off method – Similar arguments of monotonicity can be applied to ensure the endogeny of the initial process. The key variation comes from noticing that the cut-offs that are made in the past may be done at different values for different times.

First, for the simplicity of the exposition, we will impose additional assumptions. Namely, assume that

- the dimension d is finite;
- the relation function R is continuous in X_1, \dots, X_d ;
- the relation function takes finite values whenever all its arguments are finite.

All these assumptions hold in the setting of random metrics (Section 5) and the exposition under these assumptions clarifies the main ideas. However, they are unnecessary for establishing our main theorem, and we will remove them at the end of this section; see Proposition 4.

Consider any sequence $\mathbf{A} = \{a_n\}_{n=1}^\infty$ of cut-off values $a_n \in (-\infty, +\infty]$, and the corresponding sequence of relation functions R_{a_n} (where $R_{+\infty}$ is nothing else but the initial recursion relation R). We are going to construct (cf. [9, § 4.3]) a random tree process $\{(X_t^{\mathbf{A}}, \xi_t)\}_{t \in \mathcal{T}}$, whose variables verify

$$X_t^{\mathbf{A}} = R_{a_{\|t\|}}(X_{t_1}^{\mathbf{A}}, \dots, X_{t_d}^{\mathbf{A}}; \xi_t) \quad (6)$$

instead of (2); observe that the recursion relation changes with the distance to the root.

Definition 2. Let a sequence $\mathbf{A} = \{a_n\}_{n=1}^\infty$, $a_n \in (-\infty, +\infty]$, be given. For any family of values $\{\xi_t\}_{t \in \mathcal{T}}$ and any $k \in \mathbf{N}$, we recursively define $\{X_t^{\mathbf{A},k}\}_{t \in \mathcal{T}}$ as

$$\begin{cases} X_t^{\mathbf{A},k} = +\infty & \text{if } \|t\| \geq k, \\ X_t^{\mathbf{A},k} = R_{a_{\|t\|}}(X_{t_1}^{\mathbf{A},k}, \dots, X_{t_d}^{\mathbf{A},k}; \xi_t) & \text{if } \|t\| < k. \end{cases} \quad (7)$$

Directly from this definition, assumption A1), and using the assumed continuity of the function R , we have:

Proposition 1. Let $\mathbf{A} = \{a_n\}_{n=1}^\infty$, $a_n \in (-\infty, +\infty]$, be a sequence and let $\{\xi_t\}_{t \in \mathcal{T}}$ be a family of real numbers. For any fixed $t \in \mathcal{T}$, the sequence $\{X_t^{\mathbf{A},k}\}_{k=1}^\infty$ is non-increasing. In particular, the limit

$$X_t^{\mathbf{A}} := \lim_{k \rightarrow \infty} X_t^{\mathbf{A},k} \quad (8)$$

exists (though it may be infinite) and satisfies the relation (6).

When some a_n is finite, then for any $t \in \mathcal{T}$ such that $\|t\| \leq n$, the limit $X_t^{\mathbf{A}}$ is not $+\infty$. In particular, if there are infinitely many finite a_n 's, all the limits $X_t^{\mathbf{A}}$ are not $+\infty$ (though they still can be $-\infty$).

Moreover, for a fixed family $\{\xi_t\}_{t \in \mathcal{T}}$, this limit is monotone on \mathbf{A} : if for $\mathbf{A} = \{a_n\}$ and $\mathbf{A}' = \{a'_n\}$ we have $a_n \leq a'_n$ coordinate-wise then for any $t \in \mathcal{T}$, we have $X_t^{\mathbf{A}} \leq X_t^{\mathbf{A}'}$.

When the family $\{\xi_t\}_{t \in \mathcal{T}}$ is considered as a family of random variables (i.i.d. of law m), the definition (7) makes the $X_t^{\mathbf{A},k}$'s random variables. By the symmetry in the construction, all the laws of the $X_t^{\mathbf{A},k}$'s coincide at any fixed depth $\|t\| = n$; we denote this law by $\mu_n^{\mathbf{A},k}$. It follows from (7) that these laws satisfy (the non-stationary) RDEs:

$$\begin{cases} \mu_k^{\mathbf{A},k} = \text{Dirac}_{+\infty}, \\ \mu_n^{\mathbf{A},k} = \Phi_{a_n}[\mu_{n+1}^{\mathbf{A},k}] & \text{if } n < k. \end{cases} \quad (9)$$

Passing to the limit in (9), as $k \rightarrow \infty$, we get that the laws $\mu_{\|t\|}^{\mathbf{A}}$'s of the $X_t^{\mathbf{A}}$'s from Proposition 1 also satisfy the RDEs

$$\mu_n^{\mathbf{A}} = \Phi_{a_n}[\mu_{n+1}^{\mathbf{A}}]. \quad (10)$$

By construction, the RTP $\{(X_t^{\mathbf{A}}, \xi_t)\}_{t \in \mathcal{T}}$ is *endogenous* (cf. [3, Lemma 15]).

3.3 Removing the cut-offs – The next step is to start *removing* the cut-offs. The non-stationary process, constructed with Proposition 1, is endogenous, but the cut-offs alter the recurrence relation and the distribution of the X_t 's. The second step is to start removing the cut-offs one by one, starting from above, and thus constructing a new family of endogenous RTPs (this time an increasing one). Explicitly:

Definition 3. Let a sequence $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$, $a_n \in (-\infty, +\infty]$, be given. Define a family of new sequences $\mathbf{A}^{(\ell)} = \{a_n^{(\ell)}\}$ by

$$a_n^{(\ell)} = \begin{cases} a_n, & \text{if } n \geq \ell, \\ +\infty, & \text{if } n < \ell. \end{cases}$$

The RDEs (10) for the new sequences are

$$\mu_n^{\mathbf{A}^{(\ell)}} = \begin{cases} \Phi_{a_n}[\mu_{n+1}^{\mathbf{A}^{(\ell)}}], & \text{if } n \geq \ell, \\ \Phi[\mu_{n+1}^{\mathbf{A}^{(\ell)}}], & \text{if } n < \ell. \end{cases} \quad (11)$$

Notice that when $n < \ell$, one has

$$\mu_n^{\mathbf{A}^{(\ell)}} = \Phi^{\ell-n}[\mu_{\ell}^{\mathbf{A}^{(\ell)}}]. \quad (12)$$

The following proposition is deduced from the monotonicity statement in Proposition 1, and again from the continuity assumption for the function R :

Proposition 2. Let $\mathbf{A} = \{a_n\}_{n=1}^{\infty}$, $a_n \in (-\infty, +\infty]$, be a sequence and let $\{\xi_t\}_{t \in \mathcal{T}}$ be a family of real numbers. For any fixed $t \in \mathcal{T}$, the sequence $\{X_t^{\mathbf{A}^{(\ell)}}\}_{\ell=1}^{\infty}$ is non-decreasing. In particular the limit

$$\tilde{X}_t^{\mathbf{A}} := \lim_{\ell \rightarrow \infty} X_t^{\mathbf{A}^{(\ell)}} \quad (13)$$

exists, and satisfies the recurrence relation

$$\tilde{X}_t^{\mathbf{A}} = R(\tilde{X}_{t_1}^{\mathbf{A}}, \dots, \tilde{X}_{t_d}^{\mathbf{A}}; \xi_t) \quad (14)$$

Observe that the limits (13) can possibly be infinite, even if we assume that there are infinitely many finite a_n 's.

As above, when the family $\{\xi_t\}_{t \in \mathcal{T}}$ is considered as a family of random variables (i.i.d. of law m), the definition (13) makes the $\tilde{X}_t^{\mathbf{A}}$'s random variables. For any $k \in \mathbb{N}$, the limit

$$\tilde{\mu}_k^{\mathbf{A}} = \lim_{\ell \rightarrow \infty} \mu_k^{\mathbf{A}^{(\ell)}} \quad (15)$$

is the law of the $\tilde{X}_t^{\mathbf{A}}$'s with $\|t\| = k$. They satisfy the RDEs

$$\tilde{\mu}_k^{\mathbf{A}} = \Phi[\tilde{\mu}_{k+1}^{\mathbf{A}}],$$

though they can be trivial solutions (namely $\tilde{\mu}_k^{\mathbf{A}} = \text{Dirac}_{+\infty}$ or $\tilde{\mu}_k^{\mathbf{A}} = \text{Dirac}_{-\infty}$).

Nonetheless, by construction, for any sequence \mathbf{A} , the process $\{\tilde{X}_t^{\mathbf{A}}\}_{t \in \mathcal{T}}$ is *endogenous*. Hence, Theorem 1 will be proved – under our additional assumptions – as soon as we establish the following:

Proposition 3. *Let $\bar{\mu}$ be a solution to a RDE*

$$\bar{\mu} = \Phi[\bar{\mu}], \quad (16)$$

given by the assumption A4), where the recurrence relation R and the measure m satisfy all the assumptions A1)–A7). Then there exists a sequence \mathbf{A} such that $\tilde{\mu}_n^{\mathbf{A}} = \bar{\mu}$ for any n .

Moreover, as a technical conclusion, for the constructed sequence \mathbf{A} we have $\mu_n^{\mathbf{A}^{(\ell)}} \in \mathcal{Q}$ for any n and ℓ .

Roughly speaking, to prove Proposition 3, we have to choose a sequence $\mathbf{A} = \{a_n\}$ of cut-offs such that any tail $\{a_n\}_{n=\ell+1}^\infty$ suffices to bring initially-infinite values to the core, while the initial part $\{a_n\}_{n=1}^\ell$ should not push it too far towards $-\infty$.

Before getting into the proof of Proposition 3, we conclude this paragraph by explaining how to get rid of the continuity assumptions (that could be a problem when there is an infinite number of variables). In particular the reader will see how the last conclusion of Proposition 3 is employed (this is the only step where we need it). Namely, we have the following:

Proposition 4. *Under the assumptions A1)–A7) the following statements hold:*

- *Assume that the sequence \mathbf{A} is such that all the measures $\mu_n^{\mathbf{A}}$'s belong to the domain \mathcal{Q} . Then the relation (6) is satisfied almost surely.*
- *Assume that the sequence \mathbf{A} is such that all the measures $\mu_n^{\mathbf{A}^{(\ell)}}$, $\tilde{\mu}_n^{\mathbf{A}}$'s belong to the domain \mathcal{Q} . Then the relation (14) is satisfied almost surely.*

Proof. For any given set of values of $\{\xi_t\}$ the sequences $\{X_t^{\mathbf{A},k}\}$ are monotonous non-increasing. Hence, passing to the limit in (7) and using the monotonicity of the function R , we get

$$\begin{aligned} X_t^{\mathbf{A}} &= \lim_{k \rightarrow \infty} X_t^{\mathbf{A},k} = \lim_{k \rightarrow \infty} R_{a_{\|t\|}}(X_{t_1}^{\mathbf{A},k}, \dots, X_{t_d}^{\mathbf{A},k}; \xi_t) \geq \\ &R_{a_{\|t\|}}(\lim_{k \rightarrow \infty} X_{t_1}^{\mathbf{A},k}, \dots, \lim_{k \rightarrow \infty} X_{t_d}^{\mathbf{A},k}; \xi_t) = R_{a_{\|t\|}}(X_{t_1}^{\mathbf{A}}, \dots, X_{t_d}^{\mathbf{A}}; \xi_t). \end{aligned} \quad (17)$$

Whence $X_t^{\mathbf{A}} \geq R(X_{t_1}^{\mathbf{A}}, \dots, X_{t_d}^{\mathbf{A}}; \xi_t)$.

Now, recall that ξ_t are in fact random variables. The law of the left hand side of (17) is $\mu_{\|t\|}^{\mathbf{A}}$, the law of the right hand side is by definition $\Phi[\mu_{\|t\|+1}^{\mathbf{A}}]$, and (17) gives the stochastic comparison

$$\Phi_{a_{\|t\|}}[\mu_{\|t\|+1}^{\mathbf{A}}] \preceq \mu_{\|t\|}^{\mathbf{A}}. \quad (18)$$

Now, showing that the inequality in (17) is in fact *almost surely* an equality is equivalent to prove the equality in (18). Let $n = \|t\|$; by assumption $\mu_n^{\mathbf{A}} \in \mathcal{Q}$, and the operator Φ_{a_n} is continuous at $\mu_{n+1}^{\mathbf{A}}$. As $\mu_n^{\mathbf{A},k} \rightarrow \mu_n^{\mathbf{A}}$ as $k \rightarrow \infty$, and we have $\mu_n^{\mathbf{A},k} = \Phi_{a_n}[\mu_{n+1}^{\mathbf{A},k}]$ for any $k > n$, passing to the limit we get the desired equality $\mu_n^{\mathbf{A}} = \Phi_{a_n}[\mu_{n+1}^{\mathbf{A}}]$.

The proof of the second part of the proposition goes in the same way, with the only difference that the sequence $X_n^{\mathbf{A}^{(\ell)}}$ is now increasing, so that the inequalities are reversed. \square

Proposition 4 shows that even without the additional assumptions, the family of random variables $\{\tilde{X}_t\}_{t \in \mathcal{T}}$, constructed using the sequence \mathbf{A} from Proposition 3, almost surely satisfies the recurrence relation (14). This, together with the endogeneity of the family \tilde{X}_t , implies Theorem 1.

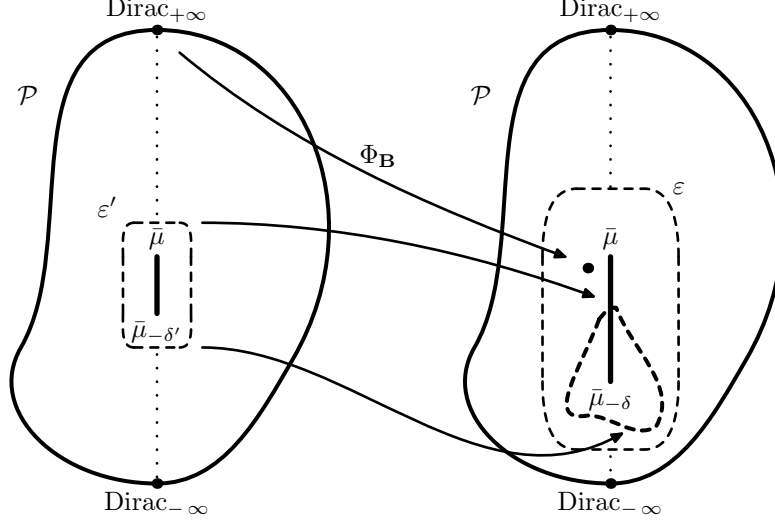


Figure 1: Properties of a $(\delta, \delta'; \varepsilon, \varepsilon')$ -block \mathbf{B} .

3.4 Blocks of cut-offs – Recall from assumption A3) that we have a metric d_Q on the space \mathcal{Q} of probability measures; all distances are considered with respect to this metric. The construction of the desired sequence \mathbf{A} will be done by blocks:

Definition 4. A *block* \mathbf{B} is a finite sequence $b_1, \dots, b_n \in (-\infty, +\infty]$. Associated with a block, we define the *block operator* $\Phi_{\mathbf{B}} : \mathcal{P} \rightarrow \mathcal{P}$ which is the composition of the cut-off operators

$$\Phi_{\mathbf{B}} = \Phi_{b_1} \circ \dots \circ \Phi_{b_n}.$$

The integer n is called the *size* of the block \mathbf{B} and is denoted by $|\mathbf{B}|$.

The blocks that we choose should work together, so let us quantify their properties. Namely, for each block we will specify how it handles “as an input”, on the one hand, the Dirac measure at infinity (the result should be close to $\bar{\mu}$), and on the other hand, a measure close to $\bar{\mu}$. To do so, we introduce the following:

Definition 5. Given $\delta > \delta' > 0$, $\varepsilon, \varepsilon' > 0$, we say that a block \mathbf{B} is a $(\delta, \delta'; \varepsilon, \varepsilon')$ -block if

- the image $\Phi_{\mathbf{B}}[\text{Dirac}_{+\infty}]$ is ε -close to $\bar{\mu}$,
- for any $c' \in (-\delta', 0)$ and any $\mu \in \mathcal{Q}$ which is ε' -close to $\bar{\mu}_{c'}$, there exists $c \in (-\delta, 0)$ such that the image $\Phi_{\mathbf{B}}[\mu]$ is ε -close to the measure $\bar{\mu}_c$ (see Fig. 1).

Observe that for arbitrary choices of $\delta, \delta', \varepsilon, \varepsilon'$, $(\delta, \delta'; \varepsilon, \varepsilon')$ -blocks may not exist (for instance, one should expect non-existence when $\varepsilon \ll \varepsilon'$). On the other hand, any $(\delta, \delta'; \varepsilon, \varepsilon')$ -block is automatically a $(\delta, \delta'; \varepsilon, \varepsilon'')$ -block for any $\varepsilon'' < \varepsilon'$.

As we shall explain in the next paragraph, Proposition 3 is a consequence of the following:

Proposition 5. Under the assumptions A1)–A7) the following statement holds: for any $\delta > \delta' > 0$, $\varepsilon > 0$, there exist $\varepsilon' > 0$ and a $(\delta, \delta'; \varepsilon, \varepsilon')$ -block.

3.5 Proposition 5 implies Proposition 3 – Take ε_1 sufficiently small so that the $2\varepsilon_1$ -neighbourhood of $\bar{\mu}$ belongs to \mathcal{Q}_0 and choose $\delta_0 > 0$ so that for any $c \in (-\delta_0, 0)$ we have $d_{\mathcal{Q}}(\bar{\mu}_c, \bar{\mu}) < \varepsilon_1$. Consider any sequence $\delta_n \in (0, \delta_0)$ that decreases to 0 (for instance $\delta_n = 2^{-n}\delta_0$) and define recursively a sequence ε_n in such a way that $\varepsilon_n \leq \frac{\varepsilon_1}{n}$ and such that there exists a $(\delta_{n-1}, \delta_n; \varepsilon_{n-1}, \varepsilon_n)$ -block \mathbf{B}_n . Let us check that the sequence \mathbf{A} , obtained by juxtaposing the blocks \mathbf{B}_n , satisfies the conditions of Proposition 3. To fix notations, we write $\mathbf{A} = \mathbf{B}_1\mathbf{B}_2\cdots$ and $s_n = |\mathbf{B}_1| + \dots + |\mathbf{B}_n|$.

As both limits (8) and (13) exist (possibly infinite), we can safely define the $X_t^{\mathbf{A}}$'s and the $\tilde{X}_t^{\mathbf{A}}$'s by considering only the limits along subsequences. We use this remark to be able to apply blocks of cut-offs entirely.

Lemma 1. *For any $n < k$ there exists $c_{n,k} \in (-\delta_n, 0)$ such that $\mu_{s_n}^{\mathbf{A}, s_k}$ is ε_n -close to $\bar{\mu}_{c_{n,k}}$.*

Proof. By (9), we have

$$\mu_{s_{n-1}}^{\mathbf{A}, s_k} = \Phi_{\mathbf{B}_n} \cdots \Phi_{\mathbf{B}_k}[\text{Dirac}_{+\infty}].$$

The measure $\mu_{s_k}^{\mathbf{A}, s_{k-1}} = \Phi_{\mathbf{B}_k}[\text{Dirac}_{+\infty}]$ is ε_{k-1} -close to $\bar{\mu}$, since \mathbf{B}_k is a $(\delta_{k-1}, \delta_k; \varepsilon_{k-1}, \varepsilon_k)$ -block.

Therefore we can prove the statement by (backwards) induction on n , taking $n = k - 1$ as the base of the induction, and with the induction step provided by the second part of Definition 5. \square

Passing to the limit as $k \rightarrow \infty$ in the conclusion of Lemma 1, we see that $\mu_{s_n}^{\mathbf{A}}$ is ε_n -close to $\bar{\mu}_{c_n}$, for some $c_n \in [-\delta_n, 0)$. Now, considering $\mu_{s_n}^{\mathbf{A}^{(s_\ell)}}$ and recalling the RDEs (11), we get the same conclusion for $n \geq \ell$, while for $n < \ell$ we get (see (12))

$$\mu_{s_n}^{\mathbf{A}^{(s_\ell)}} = \Phi^{s_\ell - s_n}[\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}].$$

The measure $\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}$ is ε_ℓ -close to $\bar{\mu}_{c_\ell}$, with $c_\ell \in (-\delta_0, 0)$ and $\varepsilon_\ell \leq \varepsilon_1$; after the choices for ε_1 and δ_0 , the triangular inequality guarantees that the measure $\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}$ belongs to \mathcal{Q}_0 :

$$d_{\mathcal{Q}}(\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}, \bar{\mu}) \leq d_{\mathcal{Q}}(\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}, \bar{\mu}_{c_\ell}) + d_{\mathcal{Q}}(\bar{\mu}_{c_\ell}, \bar{\mu}) < 2\varepsilon_1.$$

Now, recall that due to the assumption A6) the powers Φ^k converge uniformly on \mathcal{Q}_0 ; also, the measures $\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}$ converge to $\bar{\mu}$ as $\ell \rightarrow \infty$. For a uniformly convergent family of continuous mappings, the limit of the images of a convergent family of points can be calculated by substituting the limit point to the limit (continuous) map. Therefore for any n one has

$$\lim_{\ell \rightarrow \infty} \mu_{s_n}^{\mathbf{A}^{(s_\ell)}} = \lim_{\ell \rightarrow \infty} \Phi^{s_\ell - s_n}[\mu_{s_\ell}^{\mathbf{A}^{(s_\ell)}}] = \bar{\mu}.$$

Hence, recalling (15), we have the equality $\tilde{\mu}_{s_n}^{\mathbf{A}} = \bar{\mu}$ for any n , and thus $\tilde{\mu}_n^{\mathbf{A}} = \bar{\mu}$ for any n .

Finally, let us verify the technical assumption in Proposition 3. Namely, we get $\mu_{s_n}^{\mathbf{A}^{(s_\ell)}, s_k} \in \mathcal{Q}_0$ for any $k > n > \ell$; due to the compactness of \mathcal{Q}_0 we have $\mu_{s_n}^{\mathbf{A}^{(s_\ell)}} = \lim_{k \rightarrow \infty} \mu_{s_n}^{\mathbf{A}^{(s_\ell)}, s_k} \in \mathcal{Q}_0$. Finally, the invariance of \mathcal{Q} under all the Φ_a 's implies that $\mu_n^{\mathbf{A}^{(\ell)}} \in \mathcal{Q}$ for every n and ℓ .

This ends the proof of Proposition 3, assuming Proposition 5.

4 Proposition 5: constructing the blocks

4.1 Subblocks – The most delicate part of this work is to construct the desired blocks. The blocks we are looking for will be composed of elementary subblocks:

Definition 6. A *subblock* \mathbf{b} is a block such that only the last value $b_{|\mathbf{b}|}$ is finite: for every $1 \leq k \leq |\mathbf{b}| - 1$, $b_k = \infty$. We denote by $\mathbf{b}(a, \ell)$ the subblock defined by $b_{|\mathbf{b}|} = a$, $|\mathbf{b}| = \ell + 1$. So we have that the subblock operator

$$\Phi_{\mathbf{b}(a, \ell)} = \Phi^\ell \circ \Phi_a$$

is the composition of one cut-off operator and ℓ iterations of the operator Φ .

4.2 Long subblocks and centers – The convergence assumption A5) ensures that for a given initial measure $\mu \in \mathcal{Q}$, if we take a very long subblock $\mathbf{b}(a, \ell)$ (that is, with ℓ sufficiently large), then the image $\Phi_{\mathbf{b}(a, \ell)}[\mu]$ will be close to $\bar{\mu}_{\tilde{c}(\Phi_a[\mu])}$, and actually the former converges to the latter as ℓ goes to ∞ .

When sufficiently long subblocks are applied consecutively, we would like to describe the joint result of their application. The application of each of them leaves us with a measure close to some $\bar{\mu}_c$. This leads to the study the dynamics of the centers c : by the continuity of the function \tilde{c} , applying a sufficiently long subblock $\mathbf{b}(a, \ell)$ to a measure μ close to a stationary measure $\bar{\mu}_c$, we obtain a measure that is close to the stationary measure $\bar{\mu}_{\tilde{c}(\Phi_a[\bar{\mu}_c])}$. This motivates the following:

Definition 7 (Center function). For any $a \in \mathbf{R}$, we define the function $S_a : \mathbf{R} \rightarrow \mathbf{R}$ by

$$S_a(c) = \tilde{c}(\Phi_a[\bar{\mu}_c]),$$

In other words, the function S_a locates the center of the measure that we obtain by applying long subblock operators $\Phi_{\mathbf{b}(a, \ell)}$ to $\bar{\mu}_c$.

This definition naturally extends by continuity to $c = +\infty$, by putting $S_a(+\infty) = \tilde{c}(\text{Dirac}_a)$. For a given $a \in \mathbf{R}$, considering S_a as a self-map of \mathbf{R} , we write S_a^n to denote the n -fold composition $S_a \circ \dots \circ S_a$.

Remark 8. As a consequence of the assumption A1) that the operator Φ preserves the stochastic order, $S_a(c)$ is an increasing function of the cut-off value a . Also by the continuity of the center function (assumption A6)) the center $S_a(c)$ tends to c as a goes to $+\infty$.

Sometimes it will be easier to work with the *displacement of centers*, rather than with their new location:

Definition 8 (Displacement function). For any $a \in \mathbf{R}$, we define the *displacement function* $\Delta_a : \mathbf{R} \rightarrow (0, +\infty)$ by

$$\Delta_a(c) = c - S_a(c). \tag{19}$$

Notice that the displacement function takes positive values only because of Remark 8. Moreover, the same remark implies that $\Delta_a(c)$ is a decreasing function of a . We can be more precise:

Lemma 2. For any $a, c \in \mathbf{R}$, we have $\Delta_a(c) = \Delta_{a-c}(0)$. In particular $\Delta_a(c)$ is increasing with respect to c and decreasing with respect to a .

Proof. The first statement follows from the translation-equivariance A2). The second one is a corollary of the first one and of Remark 8. \square

Observe that

$$\Delta_a(0) = -S_a(0) = -\tilde{c}(\Phi_a[\bar{\mu}]),$$

therefore, as function of a , $\Delta_a(0)$ is exactly the function Δ defined in assumption A7). From the previous lemma, we have $\Delta_a(c) = \Delta(a - c)$.

4.3 Sketch of construction – The desired $(\delta, \delta'; \varepsilon, \varepsilon')$ -block \mathbf{B} will be defined by a juxtaposition of subblocks constructed in the following way. First, we apply a subblock $\mathbf{b}_0 = \mathbf{b}(b_0, \ell_0)$ with the following properties:

- 1) the cut-off value b_0 is chosen sufficiently large so that it almost does not affect the measures $\bar{\mu}_c$'s, with $c \in [-\delta', 0]$;
- 2) the number of iterations ℓ_0 is chosen sufficiently large, so that, after cutting off the Dirac measure $\text{Dirac}_{+\infty}$ at b_0 , $\Phi_{\mathbf{b}_0}$ almost sends it to $\bar{\mu}_{c_0}$, with $c_0 = \tilde{c}(\text{Dirac}_{b_0}) = S_{b_0}(+\infty)$.

After this, we choose a second subblock $\mathbf{b}_1 = \mathbf{b}(b_1, \ell_1)$ that we repeat (identically) sufficiently many times N , so that the corresponding transformation of \mathcal{P} will bring $\bar{\mu}_{c_0}$ sufficiently close to $\bar{\mu}$, and its “side effect”, when applying it to $\bar{\mu}_c$, with $c \in (-\delta', 0)$, stays under control (these measures will not be pushed to the left farther than $\bar{\mu}_{-\delta}$). Making all of this work will provide us with the desired block

$$\mathbf{B} = \underbrace{\mathbf{b}_1 \cdots \mathbf{b}_1}_{N \text{ times}} \mathbf{b}_0;$$

see Fig. 2 for how its application should treat the centers of measures.

4.4 Displacement of the centers – We shall start by choosing the cut-off values b_0 and b_1 for the subblocks and the number N of repetitions of \mathbf{b}_1 , using the approximation by the dynamics of the centers. Then, in the next paragraph, we will choose the appropriate sizes ℓ_0, ℓ_1 , concluding the proof of Proposition 5.

First, choose and fix δ'' such that $\delta > \delta'' > \delta'$. Note that as $b \rightarrow +\infty$, we have

$$S_b(-\delta') \nearrow -\delta'$$

by Remark 8, and

$$\tilde{c}(\text{Dirac}_b) \rightarrow +\infty,$$

for $\tilde{c}(\text{Dirac}_b) = b + \tilde{c}(\text{Dirac}_0)$.

Thus, there exists a sufficiently large b_0 such that $S_{b_0}(-\delta') \in (-\delta'', -\delta')$ and $\tilde{c}(\text{Dirac}_{b_0}) > 0$. We choose and fix such b_0 , and, as above, we write

$$c_0 := \tilde{c}(\text{Dirac}_{b_0}).$$

As a result of these choices, once we also choose (at the very end, § 4.5) ℓ_0 sufficiently large, we shall get that:

- 1) the measure $\Phi_{\mathbf{b}(b_0, \ell_0)}[\text{Dirac}_{+\infty}] = \Phi^{\ell_0}[\text{Dirac}_{b_0}]$ is *sufficiently close* to $\bar{\mu}_{c_0}$;
- 2) for every $c \in (-\delta', 0)$, the measure $\Phi_{\mathbf{b}(b_0, \ell_0)}[\bar{\mu}_c]$ is *sufficiently close* to one of the measures $\bar{\mu}_{c'}$'s, with $c' \in (-\delta'', 0)$.

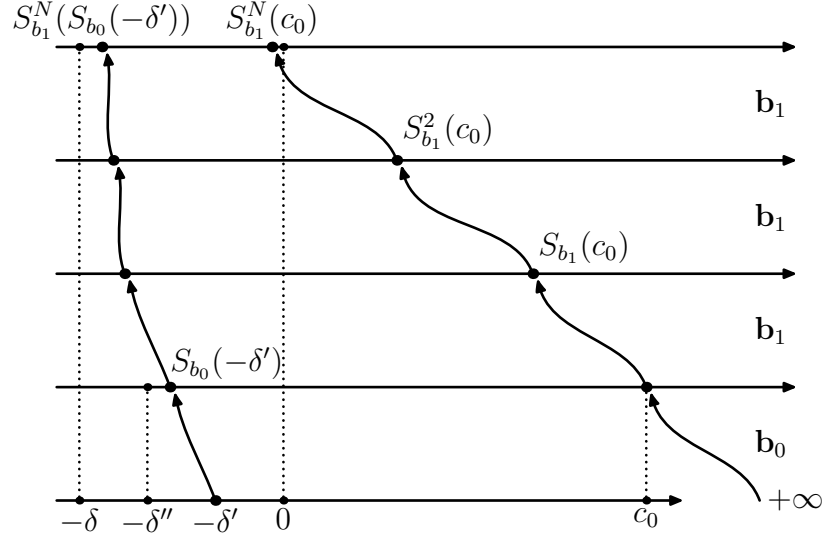


Figure 2: Construction of the block \mathbf{B} : choices for b_1 and N .

Next, let us choose the second cut-off value b_1 and the number N of repetitions for \mathbf{b}_1 . To do so, we choose and fix ε_0 such that the measures $\bar{\mu}_c$'s, with $|c| < \varepsilon_0$ are $\varepsilon/2$ -close to $\bar{\mu}$, with respect to our metric d_Q on the space \mathcal{Q} . We shall prove the following:

Lemma 3. *There exist b_1 and N such that*

- 1') $S_{b_1}^N(c_0)$ is ε_0 -close to zero, and
- 2') $S_{b_1}^N(-\delta'') \in (-\delta, -\delta')$.

This is exactly what we need for constructing the block (Definition 5): condition 1') ensures that the measure $\Phi_{\mathbf{B}}[\text{Dirac}_{+\infty}] = \Phi_{\mathbf{b}_1}^N \Phi_{\mathbf{b}_0}[\text{Dirac}_{+\infty}]$ is close to $\bar{\mu}$, while condition 2') corresponds to the second one in Definition 5.

Proof. For any b , the map $c \mapsto S_b(c)$ has no fixed points on the real line, whence

$$S_b^n(c_0) \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

In particular, for some number n of iterations, we have $S_b^n(c_0) < 0$. Take N to be the first such iteration (thus N depends on b , not chosen yet!); that is, for any given $b \in \mathbf{R}$ define

$$N(b) = \min\{n \mid S_b^n(c_0) \leq 0\}. \quad (20)$$

With this choice,

$$0 \geq S_b^{N(b)}(c_0) = S_b(S_b^{N(b)-1}(c_0)) > S_b(0), \quad (21)$$

where in the last inequality we have used that the function S_b is increasing. On the other hand, $S_b(0) \nearrow 0$ as $b \rightarrow +\infty$, hence there exists b'_1 such that $S_{b'_1}(0) \in (-\varepsilon_0, 0)$. Again by monotonicity of the center function, for any $b > b'_1$ we have

$$0 \geq S_b^{N(b)}(c_0) > S_b(0) > S_{b'_1}(0) > -\varepsilon_0,$$

thus fulfilling condition 1').

Now, we ought to choose $b_1 > b'_1$ in such a way that condition 2') is also fulfilled. To do so, note that for any b, n and any initial center $c \in \mathbf{R}$, we have a telescopic sum:

$$c - S_b^n(c) = (c - S_b^1(c)) + \cdots + (S_b^{n-1}(c) - S_b^n(c)) = \sum_{j=0}^{n-1} \Delta_b(S_b^j(c)).$$

After Lemma 2, the function Δ is monotone, thus we control the range of $c - S_b^n(c)$:

$$n\Delta_b(c) > c - S_b^n(c) > n\Delta_b(S_b^{n-1}(c)). \quad (22)$$

The inequalities (21) and (22) determine an upper bound for $N = N(b)$: by definition $S_b^N(c_0) \leq 0 < S_b^{N-1}(c_0)$, and hence (provided $b \geq b'_1$)

$$c_0 + \varepsilon_0 > c_0 - S_b^N(c_0) > N\Delta_b(S_b^{N-1}(c_0)) > N\Delta_b(0).$$

We then have

$$N < \frac{c_0 + \varepsilon_0}{\Delta_b(0)}. \quad (23)$$

From the first inequality in (22) we also have a first lower bound for $S_b^N(-\delta'')$: for any $b \in \mathbf{R}$,

$$S_b^N(-\delta'') > -\delta'' - N\Delta_b(-\delta''). \quad (24)$$

Joining the estimates in (23) and (24), for any $b \geq b'_1$ one has

$$S_b^N(-\delta'') > -\delta'' - (c_0 + \varepsilon_0) \frac{\Delta_b(-\delta'')}{\Delta_b(0)}.$$

To establish conclusion 2'), it suffices thus to find $b_1 > b'_1$ such that

$$-\delta'' - (c_0 + \varepsilon_0) \frac{\Delta_{b_1}(-\delta'')}{\Delta_{b_1}(0)} > -\delta, \quad (25)$$

or, in other words, such that

$$\frac{\Delta(b_1 + \delta'')}{\Delta(b_1)} = \frac{\Delta_{b_1}(-\delta'')}{\Delta_{b_1}(0)} < \frac{\delta - \delta''}{c_0 + \varepsilon_0}.$$

Under our assumption A7), the existence of such a b_1 is guaranteed. Indeed, set

$$\frac{\delta - \delta''}{c_0 + \varepsilon_0} = \frac{1}{C}.$$

Assuming the contrary, we would have that for any $a \geq b'_1$,

$$\frac{\Delta(a + \delta'')}{\Delta(a)} \geq \frac{1}{C}; \quad (26)$$

fixing any $a_0 > b'_1$, we have the inequality (26) for every $a = a_0 + k\delta''$, $k = 0, 1, \dots$. Thus, for any $m \in \mathbf{N}$, one has

$$\frac{\Delta(a_0 + m\delta'')}{\Delta(a_0)} = \prod_{k=0}^{m-1} \frac{\Delta(a_0 + (k+1)\delta'')}{\Delta(a_0 + k\delta'')} \geq \frac{1}{C^m}.$$

So, for any $m \in \mathbf{N}$, one has $\Delta(a_0 + m\delta'') \geq \Delta(a_0) \cdot C^{-m}$. However this contradicts the assumption A7) of superexponential decrease of Δ (for $\beta = \frac{\log C}{\delta''}$). This completes the proof of the lemma. \square

4.5 Returning to the space of measures – The initial cut-off value b_0 has already been chosen. Now we choose and fix b_1 and N satisfying the conclusions of Lemma 3. We are ready to conclude the proof of Proposition 5: we need to determine the lengths of the blocks \mathbf{b}_0 and \mathbf{b}_1 in order to have a good approximation of the dynamics with the center function.

We will need the following easy lemma, whose proof is left to the reader:

Lemma 4. *Let $U_{\mathcal{F}}, U_{\mathcal{G}} \subset \mathcal{P}$ be two open sets, and let $\mathcal{F}_n : U_{\mathcal{F}} \rightarrow \mathcal{P}$, $\mathcal{G}_n : U_{\mathcal{G}} \rightarrow \mathcal{P}$ be two sequences of continuous maps, which converge to functions \mathcal{F} and \mathcal{G} (resp.), uniformly on $U_{\mathcal{F}}$ and $U_{\mathcal{G}}$ respectively. Let $\mathcal{K} \subset U_{\mathcal{G}}$ be a compact set such that $\mathcal{G}(\mathcal{K}) \subset U_{\mathcal{F}}$. Then there exist n_0 and an open neighbourhood U of \mathcal{K} in $U_{\mathcal{G}}$ such that, for all $n \geq n_0$, the composition $\mathcal{F}_n \circ \mathcal{G}_n$ is defined on U and converges to $\mathcal{F} \circ \mathcal{G}$ uniformly on U , as $n \rightarrow \infty$.*

Fix the space \mathcal{Q} and the compact neighbourhood \mathcal{Q}_0 satisfying the assumptions A5) and A6). Consider the compact segment

$$\mathcal{K}_0 := \{\bar{\mu}_c \mid c \in [-\delta', 0]\}.$$

Up to taking a larger \mathcal{Q}_0 , we can suppose that it contains \mathcal{K}_0 and the cut-off measures $\Phi_{b_0}[\mathcal{K}_0]$. Due to the assumptions A5) and A6), given $\mu \in \mathcal{Q}$, one has the convergence

$$\lim_{\ell \rightarrow \infty} \Phi^\ell[\mu] = \bar{\mu}_{\tilde{c}(\mu)},$$

hence the operators $\{\Phi_{\mathbf{b}(b_0, \ell)}\}_{\ell \in \mathbf{N}}$ pointwise converge on \mathcal{Q} to the operator $\mathcal{F} : \mu \mapsto \bar{\mu}_{\tilde{c}(\Phi_{b_0}[\mu])}$ as $\ell \rightarrow \infty$, and the convergence is uniform on \mathcal{Q}_0 .

Note that on the line $\{\bar{\mu}_c\}_{c \in \mathbf{R}}$ the limit map \mathcal{F} acts by $\bar{\mu}_c \mapsto \bar{\mu}_{S_{b_0}(c)}$. Hence, considering our choice for b_0 , the image $\mathcal{F}[\mathcal{K}_0]$ is contained in the segment

$$\mathcal{K}_1 := \{\bar{\mu}_c \mid c \in [-\delta'', 0]\}.$$

Up to extending the neighbourhood \mathcal{Q}_0 , we can suppose that it contains \mathcal{K}_1 . Using again the assumptions A5) and A6), given $\mu \in \mathcal{Q}$, one has the convergence

$$\lim_{\ell \rightarrow \infty} \Phi_{\mathbf{b}(b_1, \ell)}[\mu] = \bar{\mu}_{\tilde{c}(\Phi_{b_1}[\mu])},$$

hence the operators $\Phi_{\mathbf{b}(b_1, \ell)}$ converge to the operator $\mathcal{G} : \mu \mapsto \bar{\mu}_{\tilde{c}(\Phi_{b_1}[\mu])}$ as $\ell \rightarrow \infty$, and the convergence is uniform on \mathcal{Q}_0 .

Applying Lemma 4 to every composition in

$$\mathcal{G}_\ell := \underbrace{\Phi_{\mathbf{b}(b_1, \ell)} \circ \cdots \circ \Phi_{\mathbf{b}(b_1, \ell)}}_{N \text{ times}} \circ \Phi_{\mathbf{b}(b_0, \ell)},$$

we see that there is a neighbourhood U of \mathcal{K}_0 on which \mathcal{G}_ℓ converges uniformly to $\mathcal{G}^N \circ \mathcal{F}$. Finally, by construction

$$\mathcal{G}^N \mathcal{F}[\text{Dirac}_{+\infty}] = \bar{\mu}_{S_{b_1}^N(c_0)},$$

and this measure is $\varepsilon/2$ -close to $\bar{\mu}$, because of the inequality $|S_{b_1}^N(c_0)| < \varepsilon_0$ and the choice of ε_0 . Hence, for every sufficiently large ℓ , the measure $\mathcal{G}_\ell[\text{Dirac}_{+\infty}]$ is ε -close to $\bar{\mu}$.

On the other hand, for any $c' \in [-\delta', 0]$ we have

$$\mathcal{G}^N \mathcal{F}[\bar{\mu}_{c'}] = \bar{\mu}_{c'},$$

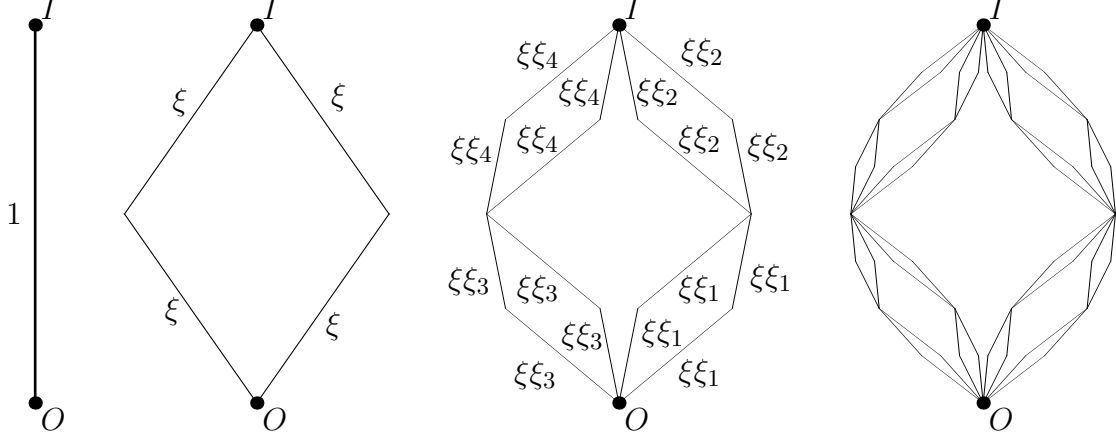


Figure 3: Multiplicative cascade on a hierarchical diamond graph.

where $c = S_{b_1}^N(S_{b_0}(c')) \in [-\delta, 0]$. Due to the uniform convergence (and hence uniform continuity), there exists $\varepsilon' > 0$ such that for every sufficiently large ℓ , for every $c' \in [-\delta, 0]$ and every $\mu' \in \mathcal{Q}_0$ that is ε' -close to $\bar{\mu}_{c'}$, we have that $\mathcal{G}_\ell[\mu']$ is ε -close to $\bar{\mu}_c$, for some $c \in [-\delta, 0]$.

Taking $\ell_0 = \ell_1 = \ell$ sufficiently large so that both conclusions above hold, we see that the block \mathbf{B} , obtained by juxtaposition of N copies of $\mathbf{b}(b_1, \ell)$ and one $\mathbf{b}(b_0, \ell)$, satisfies the conclusions of Proposition 5.

With this, the proof of Theorem 1 is accomplished.

5 Random metrics on hierarchical graphs

5.1 Background – Let us briefly recall the setting of the problem (see [9, §8.1]). Assume that we are given a graph Γ with a chosen orientation on the edges and with two marked vertices, I and O , and a law m supported on $\mathbf{R}_+ = (0, +\infty)$. One can construct a sequence of marked weighted graphs (Γ_n, I, O) in the following way:

- Γ_0 is the edge between I and O , whose length is 1.
- For each $n \geq 1$, we replace each edge e of Γ_{n-1} by a rescaled copy of the basic graph Γ ; here I is glued to the beginning of the edge, and O to the end. The length of every edge of the new copy is taken to be $\xi_e \cdot |e|$, where $|e|$ is the length of the replaced edge e , and all random variables ξ_e are i.i.d. with the law m .

This process is illustrated on Fig. 3 for the diamond graph. Each graph Γ_n is equipped with its graph-distance d_n , and the vertices of Γ_n are naturally included in the set of vertices of Γ_{n+1} . The question is whether there exists a normalization constant λ such that the sequence of distances $\lambda^n d_n$ converges to a distance (for instance, on the set which is the union of the vertices of all the Γ_n 's). At a combinatorial level, the limit space is the *hierarchical graph* associated with (Γ, I, O) .

Remark 9. An important remark is that for a generic law for the random variable ξ , with no normalization by λ , the distances obtained by this process do not converge. Indeed, if we replace the law of ξ by the law of $(1 + \varepsilon)\xi$, the distances d_n obtained on the n th step will be multiplied by $(1 + \varepsilon)^n$. Thus, if we have convergence for the initial law without a normalization by λ , the metrics obtained with any $\varepsilon > 0$ will explode, while with any $\varepsilon < 0$ will collapse.



Figure 4: The racket graph.

One can also consider the process in the other direction, saying that we are replacing each of the edges of Γ by an independent copy of Γ_{n-1} , and then rescale the result by a random common factor ξ . This process leads us to a RDE, which relates the IO -distance \hat{X} (that is, length of the shortest path between these two points) to the analogous distances \hat{X}_e 's in the children graphs:

$$\hat{X} \stackrel{d}{=} \lambda \hat{\xi} \cdot \min_{\pi} \sum_{e \in \pi} \hat{X}_e, \quad (27)$$

where the minimum is taken over all the paths π joining I to O in Γ . For instance, for the diamond graph one has

$$\hat{R}_{\lambda}(\hat{X}_1, \dots, \hat{X}_4; \hat{\xi}) = \lambda \hat{\xi} \cdot \min(\hat{X}_1 + \hat{X}_2, \hat{X}_3 + \hat{X}_4).$$

Passing to the logarithmic coordinates $X_i = \log \hat{X}_i$, $\xi = \log \lambda \hat{\xi}$, the RDE (27) turns into

$$R(X_i; \xi) = \xi + \log \min_{\pi} \sum_{e \in \pi} \exp(X_e). \quad (28)$$

These are the RDEs of Theorem 2. Note that we have incorporated the (unknown) critical drift $\log \lambda$ in the law of ξ ; its existence for non-pivotal graphs is guaranteed by [9, Thm. 1]. In particular, in the statement of Theorem 2 we are taking $\xi = \mathcal{N}(a, \sigma^2)$, where a is chosen for a given σ^2 in such a way that (28) admits a solution that is almost surely finite.

However, to use [9, Thm. 1] (and other results *ibid.*), we need to add a restriction on the geometry of the graph: we say that a graph is *non-pivotal* if there is no straight edge between I and O (a *shortcut* edge), nor edges whose removal disconnect the graph (*bridge* edges). For instance, the diamond graph is non-pivotal and, with the figure-eight graph (which was the leading example in [9]), it is the simplest graph which is non-pivotal. An example of pivotal graph is exhibited by what we call the *racket* graph, which has bridge edges (see Fig. 4).

The different behaviour of the model on pivotal and non-pivotal graphs is partly detected by the *percolation function* $\theta_{\Gamma}(p)$ that is defined as the probability that I and O are connected in Γ when considering a Bernoulli percolation of parameter p (see [9, Def. 12]). For non-pivotal graphs, $p = 0$ and $p = 1$ are super-attracting fixed points, while for pivotal graphs there is one which is (topologically) repelling.

Remark 10. This problem of finding a limit random metric was motivated by the so-called (mathematical) 2D Liouville field theory (see [9, § 3.3]). Hierarchical graph have the advantage that the problem can be formulated in the language of RDEs. An interesting recent work, closer to the original problem, has been done recently by Ding and Dunlap [5]: in some special cases (high temperatures) they show the existence of a non-trivial random metric on the square, making approximations with random metrics on discrete square grids.

5.2 Establishing assumptions – For any non-pivotal hierarchical graph, A1) is evident, as well as A2): changing X_i to $X_i + c$ multiplies the distances \hat{X}_i by e^c , that multiplies the resulting distance \hat{X} by the same constant (being a simple rescaling), and finally changes the value of R by $\log e^c = c$.

For the remaining conditions, from [9, Lemma 2 and Prop. 12] we can choose any sufficiently small $\alpha > 0$ such that α and $1 - \alpha$ belong respectively to the basins of attraction of 0 and 1 for the percolation function θ_Γ . Possibly choosing a smaller value, we suppose that for such an α , Proposition 7 below holds. We set

$$\mathcal{Q} = \{\mu \in \mathcal{P} \mid \mu(\{\pm\infty\}) \leq \alpha\}.$$

and we equip it with a metric $d_{\mathcal{Q}}$ defining the weak-* topology. It is straightforward to verify that $(\mathcal{Q}, d_{\mathcal{Q}})$ satisfy the assumption A3).

Condition A4) is given by [9, Thm. 1] (as well as by the choice of the normalization constant λ). Pointwise convergence in condition A5) is implied by [9, Thm. 2]: if the initial measures belong to $\mathcal{P}_0 \cap \mathcal{Q}$, then it is exactly the statement of that theorem, otherwise, it is enough to choose α from [9, Prop. 5] (which must satisfy A5)). For assumption A6), uniform convergence on a neighbourhood of $\bar{\mu}$ can be proved with arguments similar to those for [9, Thm. 2], as we will explain with Lemma 5 below. Finally, condition A7) will be verified with Lemma 6.

5.3 Reminder: upper and lower bounds – Before establishing uniform convergence and superexponential decrease, let us remind some of the conclusions and arguments from [9, § 6], that we will be using and extending here. First, we have the following convergence result, that is given by a part of the conclusions of [9, Thm. 2], rewritten in the logarithmic scale:

Proposition 6. *Assume that the graph Γ is non-pivotal and that the law m of ξ is given by $m = \rho(x)dx$, where $\rho : \mathbf{R} \rightarrow (0, +\infty)$ is continuous. Let $\bar{\mu} \in \mathcal{P}_0$ be a solution to the corresponding RDE. Then, for any initial measure $\mu \in \mathcal{P}_0$ there exists $c =: \tilde{c}(\mu)$ such that $\Phi^n[\mu] \rightarrow \bar{\mu}_c$ as $n \rightarrow \infty$.*

This proposition gives us the pointwise convergence needed in A5), although for measures from \mathcal{P}_0 only. However, the same arguments that were used in [9] imply the convergence in \mathcal{Q} , as well as its uniformity on some neighbourhood \mathcal{Q}_0 of $\bar{\mu}$. We recall here the main arguments. From now on, to fix notation, we shall write F_μ for the distribution function of a measure μ .

Definition 9. A measure μ is of (*upper*) *class* (α, δ) if the partition functions F_μ and $F_{\bar{\mu}}$ satisfy the inequalities

- $F_\mu(x) \leq F_{\bar{\mu}}(x)$ on $[\kappa_\alpha, \kappa_{(1-\alpha)}]$,
- $F_\mu(x) \leq F_{\bar{\mu}}(x) + \delta$ on $[-\infty, \kappa_\alpha) \cup (\kappa_{(1-\alpha)}, +\infty]$,

where κ_α and $\kappa_{1-\alpha}$ are the α - and $(1 - \alpha)$ -quantiles of the measure $\bar{\mu}$ respectively (see Fig. 5). We denote the set of such functions by $\mathcal{C}_{\alpha, \delta}$.

Note that there is no reason to consider $\delta > \alpha$: the class $\mathcal{C}_{\alpha, \delta}$ for $\delta > \alpha$ coincides with $\mathcal{C}_{\alpha, \alpha}$. We also consider the family of translation operators T_r :

Definition 10. For any $r \in \mathbf{R}$, the operator T_r is defined by sending any measure $\mu \in \mathcal{P}$ to the shifted measure $T_r[\mu] := \mu(\cdot - r)$.

Two propositions from [9] describe the iterations of measures, when starting from measures of some class (α, δ) . The first of them states that the Φ -image of a measure of class (α, δ) , after a slight translation, belongs to an even “better” class:

Proposition 7 ([9, Prop. 5]). *For every sufficiently small α , there exists a constant $L = L(\alpha) > 0$ such that for every $\delta \in (0, \alpha]$ the operator $T_{L\delta}\Phi$ sends $\mathcal{C}_{\alpha, \delta}$ in $\mathcal{C}_{\alpha, \delta/2}$.*

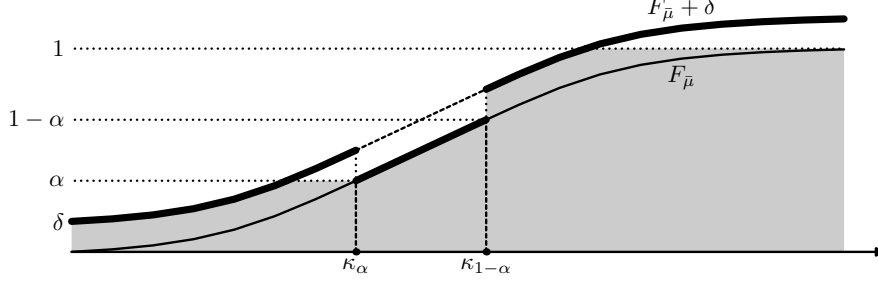


Figure 5: For a measure μ of class (α, δ) , the graph of its distribution function stays below the bold curves, and by monotonicity it is contained in the grey-filled area.

In short, a coupling argument allows to get a uniform estimate for the partition functions $F_{\Phi[\mu]}(x) \leq F_{\bar{\mu}}(x) + \text{const} \cdot \delta$; a translation by $L\delta$ then removes the correction $\text{const} \cdot \delta$ in the center (as the density of $\bar{\mu}$ is positive, in a compact domain it is bounded away from zero by some constant $\frac{1}{L}$). On the other hand, the tail of the image depends on the tail of the starting measure in a way resembling a contracting operator. Roughly speaking, the reason is that the graph is non-pivotal, so it takes at least two large (“parallel”) edges to make a large distance in the glued graph, and it takes at least two short (“consecutive”) edges to make a short distance in the glued graph: this leads to a nearly squaring (at least) of the small probabilities of these events. We refer the reader to [9, § 6.1] for details.

From now on, let us fix any α and L such as in Proposition 7. Applying this proposition inductively provides us with the following proposition:

Proposition 8 ([9, Prop. 7]). *Given a measure $\mu \in \mathcal{P}$ of class (α, δ) , for any n one has that for any $x \in \mathbf{R}$,*

$$F_{\Phi^n[\mu]}(x) \leq F_{\bar{\mu}_{-2L\delta}}(x) + \delta/2^n.$$

We recall the following definition, motivated by Proposition 8 (cf. [9, § 6.2, Def. 8])

Definition 11. A measure μ is *asymptotically upper bounded* (resp. *lower bounded*) by a translation $\bar{\mu}_c$ of the stationary measure $\bar{\mu}$, if for any $\varepsilon > 0$ there exists n_0 such that for any $n > n_0$ and $x \in \mathbf{R}$,

$$F_{\Phi^n[\mu]}(x) \leq F_{\bar{\mu}_c}(x) + \varepsilon \quad (\text{resp. } F_{\Phi^n[\mu]}(x) \geq F_{\bar{\mu}_c}(x) - \varepsilon).$$

5.4 Uniform convergence and superexponential decrease estimate – We can now show that the assumption A6) holds:

Lemma 5. *For any compact set \mathcal{K} in the space \mathcal{Q} the convergence*

$$\Phi^n[\mu] \rightarrow \bar{\mu}_{\tilde{c}(\mu)}$$

is uniform on \mathcal{K} and the function \tilde{c} is continuous on \mathcal{K} .

Proof. Take any measure $\mu \in \mathcal{K}$. By Proposition 6 (or [9, Thm. 2]) and the definition of \tilde{c} , the measures $\Phi^n[\mu]$ converge to the measure $\bar{\mu}_{\tilde{c}(\mu)}$ and hence the distribution functions of $\Phi^n[\mu]$ converge uniformly to the (continuous) distribution function of $\bar{\mu}_{\tilde{c}(\mu)}$. In particular, for an arbitrarily small $\delta > 0$ there exists n_0 such that the partition function of $\Phi^{n_0}[\mu]$ is $\delta/2$ -close to the one of $\bar{\mu}_{\tilde{c}(\mu)}$.

For what follows, let us assume $\tilde{c}(\mu) = 0$ (otherwise we translate everything by $-\tilde{c}(\mu)$ and use the translation-equivariance). Use this choice and apply again (as in Proposition 7) a translation

by $L\delta$ to handle the “core” part. The distribution function of the obtained measure $T_{L\delta}\Phi^{n_0}[\mu]$ then belongs to the class $\mathcal{C}_{(\alpha,\delta)}$. Moreover, due to the continuity of Φ^{n_0} , the same holds for any measure μ' belonging to some neighbourhood U of the initial measure μ .

The application of Proposition 8 gives that for any $n > n_0$ and for any $\mu' \in U$ the image

$$T_{L\delta}\Phi^n[\mu'] = \Phi^{n-n_0}[T_{L\delta}\Phi^{n_0}[\mu']]$$

is asymptotically upper bounded by $\bar{\mu}_{-2L\delta}$, and thus for any $n > n_0$ and $x \in \mathbf{R}$,

$$F_{\Phi^n[\mu']}(x) \leq T_{-3L\delta}[F_{\bar{\mu}}](x) + \delta/2^{n-n_0}.$$

In fact, as $\Phi^{n_0}[\mu]$ is $\delta/2$ -close to $\bar{\mu}$, working with measures in *lower classes* (α, δ) , we can repeat the same arguments to find asymptotically lower bounds (possibly decreasing the choice of α): for any $n > n_0$ and $\mu' \in U$, for any $x \in \mathbf{R}$,

$$F_{\Phi^n[\mu']}(x) \geq T_{3L\delta}[F_{\bar{\mu}}](x) - \delta/2^{n-n_0}.$$

Then passing to the limit in these upper and lower bounds, since we know that $\Phi^n[\mu']$ converges to $\bar{\mu}_{\tilde{c}(\mu')}$, we have

$$T_{3L\delta}[F_{\bar{\mu}}](x) \leq F_{\bar{\mu}_{\tilde{c}(\mu')}}(x) \leq T_{-3L\delta}[F_{\bar{\mu}}](x) \quad \text{for any } x \in \mathbf{R}$$

and thus $|\tilde{c}(\mu')| \leq 3L\delta$, which proves the continuity part of the lemma (cf. [9, Prop. 8]). Finally, we see that for every measure $\mu' \in U$, the images $\Phi^n[\mu']$, for $n \geq n_0$, stay uniformly close to the corresponding translates of $\bar{\mu}$. Together with the compactness of \mathcal{K} , this ensures the desired uniformity. \square

The next lemma guarantees the last condition A7):

Lemma 6. *The function Δ decreases superexponentially: for every $\beta > 0$,*

$$\Delta(a) = o\left(e^{-\beta a}\right) \quad \text{as } a \rightarrow +\infty.$$

Proof. According to [9, Lemma 19], the tail distribution function $f_{\bar{\mu}}(x) = 1 - F_{\bar{\mu}}(x)$ of $\bar{\mu}$ has a superexponential decrease: for any $\beta > 0$,

$$f_{\bar{\mu}}(s) = o\left(e^{-\beta s}\right) \quad \text{as } s \rightarrow +\infty.$$

Consider a cut-off measure $\Phi_a[\bar{\mu}]$ for any $a > \kappa_{(1-\alpha)}$. This measure is of class $(\alpha, f_{\bar{\mu}}(a))$. Applying Proposition 8 in the same way as in the proof of Lemma 5, we see that $|\tilde{c}(\Phi_a[\bar{\mu}])| \leq 2Lf_{\bar{\mu}}(a)$, while by definition $\Delta(a) = -\tilde{c}(\Phi_a[\bar{\mu}])$. Thus, we get the desired upper bound

$$\Delta(a) \leq 2Lf_{\bar{\mu}}(a) = o(e^{-\beta a}) \quad \text{as } a \rightarrow +\infty,$$

for any $\beta > 0$. \square

6 Mean-field optimization problems

We shall only describe briefly the combinatorial models: the references mentioned within this section provide neat expositions.

6.1 Mean-field minimal matching in pseudo-dimension q – Let $K_{n,n}$ be the complete $n \times n$ bipartite graph, with i.i.d. edge-lengths. As the name suggests, the mean-field minimal matching problem consists in describing the law of the minimal total length (or cost) for a perfect matching. Mézard and Parisi, with methods borrowed from statistical physics [12, 14], guessed the asymptotic behaviour of this random total cost. After Aldous [1], the problem can be formulated involving RDEs.

We resume notation from [3, § 7.4] (see also [2]): let a pseudo-dimension $q > 0$ be chosen. Take $\xi_1 < \xi_2 < \dots$ to be jump points of a Poisson process with intensity $x^{q-1}dx$; namely, the expectation of the number of i 's such that $\xi_i < x$, is equal to x^q/q for any $x > 0$.

The mean-field minimal matching problem is associated with the RDE [3, Eq. (94)]:

$$X \stackrel{d}{=} \min_{1 \leq i < \infty} (\xi_i - X_i). \quad (29)$$

6.2 Mean-field TSP and minimum weight k -factor – Let $k \geq 1$ be an integer. The previous mean-field optimization problem can be generalized (as done in [7]) looking at k -factors instead of perfect matchings: a k -factor is a spanning k -regular subgraph. A 1-factor is exactly a perfect matching. As above, the problem of describing (asymptotically) the law of the minimal k -factor can be formulated using RDEs: with the same notations as above the equation is

$$X \stackrel{d}{=} \min_{1 \leq i < \infty}^{[k]} (\xi_i - X_i), \quad (30)$$

where $\min^{[k]}$ stays for the k th smallest element of the set.

When $k = 2$, this RDE is also associated with the solution to the mean-field approximation to the travelling salesman problem (see [3, Eq. (95)] and [6]): this is rather intuitive, for a 2-regular spanning subgraph is union of closed non-self-intersecting loops.

6.3 Tail distribution functions: cavity equations – From statistical physics, the equations (29) and (30) take the name of *cavity equations*, after Mézard and Parisi (see for instance [13, 14]). They are sometimes expressed in terms of *tail distribution functions* and it is worth to present them in this form: it will prepare the reader for the proof of Theorem 3.

First, let $\{\xi_i\}$ be a Poisson point process with intensity $\rho(x)dx$, and $\{X_i\}$ i.i.d. random variables of law μ , independent of $\{\xi_i\}$. Then the point process $\{\xi_i - X_i\}$ is also given by a Poisson process, this time, with intensity $\tilde{\rho}(x)dx$, given by a (subtractive) convolution

$$\tilde{\rho}(x) = \int_{-\infty}^{+\infty} \rho(y+x)d\mu(y)$$

(this is the so-called “displacement theorem” [10, § 5.5], see also [2, Lemma 5]).

In particular, the number of points of this new process smaller than any given x is a Poisson random variable with parameter $\tilde{\lambda}(x) = \int_{-\infty}^x \tilde{\rho}(y)dy$. Thus, if $f(x) = \mathbf{P}(X_i > x)$ is the tail distribution function of some X_i , then the parameter of the new Poisson random variable is given by

$$\tilde{\lambda}(x) = I[f](x) := \int_{-\infty}^{+\infty} \rho(x+y)f(y)dy.$$

Next, a Poisson random variable with parameter λ is less than k with probability

$$P_k(\lambda) := e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!}, \quad (31)$$

and thus the tail distribution function of $\min_{1 \leq i < \infty}^{[k]} (\xi_i - X_i)$ is equal to $P_k(I[f](x))$.

A tail distribution function f solves the cavity equation (that is, f is the tail distribution function of a random variable solution to the RDE (30)), if and only if

$$f = P_k \circ I[f].$$

When the pseudo-dimension q is equal to 1 in (29), Aldous proved in [2] that the tail distribution function of the logistic distribution

$$f(x) = \frac{1}{1 + e^x}$$

solves the corresponding cavity equation. For other values of parameters no such explicit solution is known (and, most probably, it never admits any reasonable analytic expression).

6.4 Remarks – These distributional equations differ from those that we have studied in the previous section, and in several aspects. First, ξ is no longer a real-valued random variable, as it takes values in the space of sequences $\{\xi_i\}$. Second, there is an infinite number of X_i 's that are used. Third, the relation function R is monotone non-increasing instead of monotone non-decreasing.

The first issue is not so important: we have never used that ξ was taking real values in the proof of Theorem 1; the arguments from the proof of Theorem 2 will require some slight modifications, but there is nothing substantial to be changed. Also, as we explained in Section 3, dealing with an infinite number of variables requires some extra care, but it can be managed.

Lastly, the third issue can be easily solved by passing to the square of the map Φ , doing two iterations at once. For example, in (29), this leads to the RTP

$$X \stackrel{d}{=} \min_{1 \leq i < \infty} (\xi_i - \min_{1 \leq j < \infty} (\xi_{ij} - X_{ij})) = \min_{1 \leq i < \infty} (\xi_i + \max_{1 \leq j < \infty} (X_{ij} - \xi_{ij})). \quad (32)$$

This RTP is translation-equivariant, and it looks natural to try to apply the methods of [9]. However, one should note that its features look more like those of the RDE corresponding to a *pivotal* graph. Indeed, if a measure μ has an atom at $+\infty$ with an arbitrarily small weight, its image by the operator associated with (29) is automatically $\text{Dirac}_{-\infty}$, and its image by the one associated with (32) is $\text{Dirac}_{+\infty}$. Moreover, this dramatic collapse occurs even for measures μ whose tails at $+\infty$ do not have sufficiently good decay.

To overpass this problematic issue, the solution is to restrict to a space \mathcal{Q} of measures whose tails have a sufficiently good integrability condition, as done first in [17] and then in [7, 16]. Indeed, note that from the expression (31) we have the asymptotic equivalences

$$P_k(\lambda) = O(\lambda^{k-1} e^{-\lambda}) \quad \text{as } \lambda \rightarrow +\infty \quad (33)$$

and

$$1 - P_k(\lambda) = e^{-\lambda} \sum_{j=k}^{+\infty} \frac{\lambda^j}{j!} = O(\lambda^k) \quad \text{as } \lambda \rightarrow 0. \quad (34)$$

Thus one can obtain good tail decays on the distribution function $P_k(I[f](x))$, which corresponds to the image under the operator associated with the RDE. The tail decays also allow to define an adapted metric $d_{\mathcal{Q}}$ on \mathcal{Q} , leading to an appropriate variation of the classes $\mathcal{C}_{\alpha, \delta}$, so that the methods of [9] will work. We shall come back to this in § 6.6.

6.5 Solutions to the equations – Actually, a cut-off method has already been used in [11, 16, 19] and [7] to solve the RDEs (29) and (30) respectively. Combining these works one can state:

Theorem 4 (Shah–Salez, Wästlund, Khandwawala, Salez). *Let $q \geq 1$ and let $\{\xi_i\}$ be a Poisson point process with intensity x^{q-1} . Let $k \geq 1$ be an integer. The equation (30)*

$$X \stackrel{d}{=} \min_{1 \leq i < \infty}^{[k]} (\xi_i - X_i),$$

admits a unique solution $\bar{\mu}$. The Dirac measures $\text{Dirac}_{\pm\infty}$ and the translates $\bar{\mu}_c$'s, $c \neq 0$, of the stationary measure $\bar{\mu}$, are periodic points of order 2 for Φ , and any measure $\mu \in \mathcal{P}$ converges to one of these periodic orbits. Moreover, the measures $\bar{\mu}^a$ that are solutions to the cut-off equations

$$X \stackrel{d}{=} \min \left(\min_{1 \leq i < \infty}^{[k]} (\xi_i - X_i), a \right),$$

converge to $\bar{\mu}$ as $a \rightarrow +\infty$.

Remark 11. For pseudo-dimension $0 < q < 1$, as a first partial result, Larsson [11] proved the existence of unique, globally-attractive solutions to the cut-off equations. It would be interesting to see if our methods apply to these cases.

In the remaining part, we inspect the proofs in [7, 16, 17] and verify the assumptions A1)–A7).

6.6 Verifying assumptions – Let $q > 1$ and $k \in \mathbb{N}$ be fixed, let Φ be the operator associated to the RDE (30). Given any measure $\mu \in \mathcal{P}$, we denote by f_μ its tail distribution function.

As already noticed, if we want to stay within our setting of monotone RDEs, we have to iterate the recursion twice as in (32), and actually it is better to consider both even and odd iterations: the former are non-decreasing, the latter non-increasing. The assumptions A1)–A7) will be verified for Φ^2 , namely the squared RDE (32).

We have already mentioned that for a given measure μ its tail distribution function f_μ does not verify the integrability condition

$$\int_0^{+\infty} y^{q-1} f_\mu(y) dy < +\infty, \quad (35)$$

then applying Φ collapses the measure: $\Phi[\mu] = \text{Dirac}_{-\infty}$ and $\Phi^2[\mu] = \text{Dirac}_{+\infty}$.

We define $\widehat{\mathcal{Q}}$ to be the space of measures

$$\widehat{\mathcal{Q}} = \{\mu \in \mathcal{P}_0 \mid f_\mu \text{ verifies (35)}\}$$

and we equip it with the distance

$$d_{\mathcal{Q}}(\mu, \nu) = \sup_{x \in \mathbb{R}} |f_\mu(x) - f_\nu(x)| e^{C_q |x|}, \quad (36)$$

where C_q is some (sufficiently large) constant, depending on q , that will be fixed by Proposition 9.

With respect to the distance $d_{\mathcal{Q}}$, there are measures in $\widehat{\mathcal{Q}}$ that are at infinite distance, whence we define

$$\mathcal{Q} := \{\mu \in \widehat{\mathcal{Q}} \mid d_{\mathcal{Q}}(\mu, \bar{\mu}) < +\infty\}.$$

The study of convergence to the solution in [17, Thm. 5.1] is done for the case $k = q = 1$ for which the cut-off is not needed, but it can be adapted to the general case:

Theorem 5 (Shah–Salez). *Let $q \geq 1$ and let $\{\xi_i\}$ be a Poisson point process with intensity x^{q-1} . Let $k \geq 1$ be an integer. Let $\bar{\mu}$ be the solution to the RDE (30) and $\mu \in \mathcal{Q}$ any measure. Then there exists a constant $c = \tilde{c}(\mu) \in \mathbf{R}$ such that*

$$\Phi^{2n}[\mu] \rightarrow \bar{\mu}_c \quad \text{and} \quad \Phi^{2n+1}[\mu] \rightarrow \bar{\mu}_{-c} \quad \text{as } n \rightarrow \infty.$$

Moreover, with respect to the distance $d_{\mathcal{Q}}$, the function \tilde{c} is continuous and the convergences are uniform.

Therefore, from now on, we shall only consider measures satisfying (35) and actually after four iterations we can restrict even more the space on which we study the behaviour of Φ . Indeed, using the asymptotic equivalences (33) and (34), proceeding as in [7, Lemma 3] and [16, Prop. 3.1], we have:

Lemma 7. *Let $n \geq 4$. For any measure $\mu \in \mathcal{Q}$, the image $\Phi^n[\mu]$ has the tail decays*

$$f_{\Phi^n[\mu]}(x) = O\left(x^{q(k-1)}e^{-x^q}\right) \quad \text{as } x \rightarrow +\infty. \quad (37)$$

$$1 - f_{\Phi^n[\mu]}(x) = O\left(|x|^{qk(k-1)}e^{-k|x|^q}\right) \quad \text{as } x \rightarrow -\infty. \quad (38)$$

In particular, when $q > 1$, the stationary measure has a superexponential tail decay.

The conditions A1) and A2) are immediately verified for the square Φ^2 . The space $(\mathcal{Q}, d_{\mathcal{Q}})$ for condition A3) has been chosen and continuity of operators can be checked using the explicit expression at the level of tail distribution functions, namely

$$f_{\Phi[\mu]} = P_k \circ I[f_{\mu}]$$

(this avoids dealing with the continuity of the function R defined on the infinite product $[-\infty, \infty]^{\mathbf{N}}$). The assumption A4) follows from Theorem 4 and Lemma 7: the solutions $\bar{\mu}_c$ are supported on \mathbf{R} and clearly have integrable tails. The assumption A5) is verified after Theorem 5.

Remark 12. In fact, in [17] Shah and Salez prove the convergence for the “horizontal” displacement which, together with the superexponential tail decay, guarantees the convergence with respect to our metric $d_{\mathcal{Q}}$.

We need now to establish the assumptions A6) and A7). Proceeding in the same way as in §§ 5.3–5.4, we would like to use a series of upper and lower bounds for the distribution functions that behave nicely under the iterations (compare with Definition 9, Propositions 7 and 8), and to combine them with the decay bounds on the stationary measure $\bar{\mu}$ to establish the superexponential decay of the function Δ (cf. Lemma 6).

However, the definition for the lower class (α, δ) requires some adaptation for this case: trying to copy it directly, we stumble upon measures having atoms at $+\infty$, and as we have already discussed, such measures are sent to $\text{Dirac}_{\pm\infty}$ in one or two iterations of the map Φ . Thus, instead of changing the partition function $F_{\bar{\mu}}$ by a constant δ , the allowed difference between the partition functions will depend on the point.

That is, given $M > 0$ (that plays the role of α and that will be fixed by Proposition 9) we introduce the perturbation function

$$f_{\text{pert}}(x) = \begin{cases} e^{-C_q|x|}, & |x| \geq M \\ 0, & |x| < M, \end{cases}$$

where C_q is the constant defining the distance $d_{\mathcal{Q}}$ in (36) (that will be also fixed by Proposition 9). We then define:

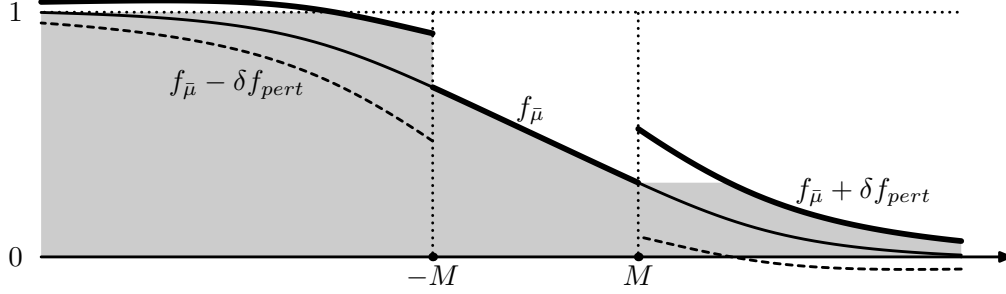


Figure 6: The adapted classes (M, δ) . The grey zone is where any tail distribution function f belonging to the adapted upper class (M, δ) is forced to stay by its monotonicity.

Definition 12. A measure $\mu \in \mathcal{Q}$ is of *adapted upper class* (M, δ) if the tail distribution function f_μ satisfies the inequality

$$f_\mu(x) \leq f_{\bar{\mu}}(x) + \delta f_{pert}(x) \quad \text{for every } x \in \mathbf{R}, \quad (39)$$

where $f_{\bar{\mu}}$ is the tail distribution function of the stationary measure $\bar{\mu}$ (see Fig. 6).

Analogously, a measure $\mu \in \mathcal{Q}$ is of *adapted lower class* (M, δ) if

$$f_\mu(x) \geq f_{\bar{\mu}}(x) - \delta f_{pert}(x) \quad \text{for every } x \in \mathbf{R}. \quad (40)$$

With this definition we can follow the strategy of [9]. We have a result analogue to Proposition 7 (from which we choose the constant C_q and M):

Proposition 9. *There exist positive constants $C_q, M, L > 0$ and $\delta_0 \in (0, 1)$ such that the following holds:*

- 1) *for any $\delta \in (0, \delta_0)$, if the measure $\mu \in \mathcal{Q}$ is of adapted upper class (M, δ) , then $T_{L\delta}\Phi[\mu]$ is of adapted lower class $(M, \delta/2)$;*
- 2) *for any $\delta \in (0, \delta_0)$, if the measure $\mu \in \mathcal{Q}$ is of adapted lower class (M, δ) , then $T_{-L\delta}\Phi[\mu]$ is of adapted upper class $(M, \delta/2)$.*

Once this proposition is proved, we proceed as in Lemma 5 to establish A6) (where \mathcal{Q}_0 can be taken, for instance, to be the closure of the $\delta_0/2$ -neighbourhood of $\bar{\mu}$), and as in Lemma 6 in order to ensure A7). The rest of this section is devoted to the proof of this proposition.

As in [9], we first find the good bounds for $f_{\Phi[\mu]}$ outside a compact set (Lemma 8), with linear (in δ) bounds on this compact set. Then we perform linear (in δ) translations to guarantee that the functions $f_{T_{\pm L\delta}\Phi[\mu]}$ verify the needed inequalities everywhere (Lemma 9). Notice that the inequalities in Lemma 8 are in the opposite directions: an upper bound for f_μ becomes a lower bound for its Φ -image, and *vice versa*.

Lemma 8. *There exist positive constants C_q, M and $K > 0$ such that for every $\delta \in (0, 1)$ the following holds:*

- 1) *if the measure $\mu \in \mathcal{Q}$ is of adapted upper class (M, δ) , then $f_{\Phi[\mu]}$ satisfies*

$$f_{\Phi[\mu]}(x) \geq \begin{cases} f_{\bar{\mu}}(x) - K\delta & \text{for every } |x| < M, \\ f_{\bar{\mu}}(x) - \frac{\delta}{2}f_{pert}(x) & \text{for every } |x| \geq M; \end{cases}$$

2) if the measure $\mu \in \mathcal{Q}$ is of adapted lower class (M, δ) , then $f_{\Phi[\mu]}$ satisfies

$$f_{\Phi[\mu]}(x) \leq \begin{cases} f_{\bar{\mu}}(x) + K\delta & \text{for every } |x| < M, \\ f_{\bar{\mu}}(x) + \frac{\delta}{2}f_{\text{pert}}(x) & \text{for every } |x| \geq M. \end{cases}$$

Lemma 9. Let C_q , M and $K > 0$ be given by Lemma 8. There exist $L > 0$ and $\delta_0 \in (0, 1)$ such that for every $\delta \in (0, \delta_0)$ the following holds:

1) if $\mu \in \mathcal{Q}$ satisfies

$$f_{\Phi[\mu]}(x) \geq \begin{cases} f_{\bar{\mu}}(x) - K\delta & \text{for every } |x| < M, \\ f_{\bar{\mu}}(x) - \frac{\delta}{2}f_{\text{pert}}(x) & \text{for every } |x| \geq M; \end{cases}$$

then $T_{L\delta}\Phi[\mu]$ is of adapted lower class $(M, \delta/2)$;

2) if $\mu \in \mathcal{Q}$ satisfies

$$f_{\Phi[\mu]}(x) \leq \begin{cases} f_{\bar{\mu}}(x) + K\delta & \text{for every } |x| < M, \\ f_{\bar{\mu}}(x) + \frac{\delta}{2}f_{\text{pert}}(x) & \text{for every } |x| \geq M. \end{cases}$$

then $T_{-L\delta}\Phi[\mu]$ is of adapted upper class $(M, \delta/2)$.

The proof of Lemma 9 is almost immediate, following the lines of [9, Lemma 17]. Namely, in the compact region $|x| < M$ the problem is handled by a translation, as the measure $\bar{\mu}$ is absolutely continuous with positive continuous density. At the same time we have

$$f_{T_{L\delta}\Phi[\mu]}(x) \geq f_{\Phi[\mu]}(x) \geq f_{-T_{L\delta}\Phi[\mu]}(x)$$

for any x , and hence the inequality for $|x| \geq M$ holds automatically.

Lemmas 8 and 9 together imply Proposition 9, and Lemma 9 is already proven. It only remains to prove Lemma 8. We start with two auxiliary estimates:

Lemma 10. For every $M > 0$, the function $I[f_{\text{pert}}](x) = \int_{-x}^{+\infty} (x+y)^{q-1} f_{\text{pert}}(y) dy$ satisfies:

$$I[f_{\text{pert}}](x) = \frac{\Gamma(q)}{C_q^q} f_{\text{pert}}(x) \quad \text{for every } x \leq -M; \quad (41)$$

and

$$I[f_{\text{pert}}](x) = O(x^{q-1}) \quad \text{as } x \rightarrow +\infty, \quad (42)$$

with the $O(x^{q-1})$ that is uniform on $M > 0$.

Proof. For $y \geq M$ we have $f_{\text{pert}}(y) = e^{-C_q y}$. Then when $x \leq -M$, we can easily compute the convolution integral after shifting the variable:

$$\begin{aligned} \int_{-x}^{+\infty} (x+y)^{q-1} e^{-C_q y} dy &= e^{C_q x} \int_0^{+\infty} y^{q-1} e^{-C_q y} dy \\ &= \frac{\Gamma(q)}{C_q^q} e^{C_q x} = \frac{\Gamma(q)}{C_q^q} f_{\text{pert}}(x), \end{aligned}$$

whence the first equality is proved. Observe that the same calculation gives that for any $|x| \leq M$,

$$\int_{-x}^{+\infty} (x+y)^{q-1} f_{pert}(y) dy = \frac{\Gamma(q)}{C_q^q} e^{-C_q M}.$$

When $x > M$, we split the convolution integral at $-M$ and find

$$\int_{-x}^{+\infty} (x+y)^{q-1} f_{pert}(y) dy = \frac{\Gamma(q)}{C_q^q} e^{-C_q M} + \int_{-x}^{-M} (x+y)^{q-1} e^{C_q y} dy.$$

The first term is a constant (and goes to zero as $M \rightarrow +\infty$). For the second term, by monotonicity we have

$$\begin{aligned} \int_{-x}^{-M} (x+y)^{q-1} e^{C_q y} dy &\leq (x-M)^{q-1} \int_{-x}^{-M} e^{C_q y} dy \\ &\leq x^{q-1} \frac{1}{C_q} (e^{-C_q M} - e^{-C_q x}) \leq \frac{e^{-C_q M}}{C_q} x^{q-1}, \end{aligned}$$

concluding the proof of the lemma. \square

Lemma 11. *For any positive λ , λ_{pert} and $\delta > 0$, the function P_k verifies*

$$P_k(\lambda + \delta \lambda_{pert}) \geq e^{-\delta \lambda_{pert}} P_k(\lambda) \quad (43)$$

and

$$P_k((\lambda - \delta \lambda_{pert})^+) \leq e^{\delta \lambda_{pert}} P_k(\lambda), \quad (44)$$

where $(\lambda - \delta \lambda_{pert})^+ = \max(\lambda - \delta \lambda_{pert}, 0)$.

The proof of Lemma 11 is a straightforward estimate. We are now ready to conclude with the proof of Lemma 8:

Proof of Lemma 8. We claim that it is enough to consider C_q so large that

$$\frac{\Gamma(q)}{C_q^q} \leq \frac{1}{4}. \quad (45)$$

We first prove 1). At the level of tail distribution functions, the operator Φ acts by $f \mapsto P_k \circ I[f]$ and reverses the order relations. We remark that the operator $f \mapsto P_k \circ I[f]$ is well-defined and order-reversing even if f is not a tail distribution function. Therefore from (39) one has the inequality

$$f_{\Phi[\mu]}(x) \geq P_k(I[f_{\bar{\mu}} + \delta f_{pert}](x)) \quad \text{for every } x \in \mathbf{R}. \quad (46)$$

Observe that the function I is defined by a convolution and is therefore linear.

Recall that $f_{\bar{\mu}}$ satisfies $f_{\bar{\mu}} = P_k \circ I[f_{\bar{\mu}}]$. Then, considering $\lambda = I[f_{\bar{\mu}}]$ and $\lambda_{pert} = I[f_{pert}]$ in Lemma 11 as functions, using (43) in (46), one has

$$\begin{aligned} f_{\Phi[\mu]}(x) &\geq e^{-\delta \lambda_{pert}(x)} f_{\bar{\mu}}(x) \\ &\geq (1 - \delta \lambda_{pert}(x)) f_{\bar{\mu}}(x) \quad \text{for every } x \in \mathbf{R}. \end{aligned}$$

Therefore, we have the inequality in 1) as soon as we prove that there exist M and $K > 0$ such that

$$\lambda_{pert}(x) f_{\bar{\mu}}(x) \leq \begin{cases} K & \text{for every } |x| < M, \\ \frac{1}{2} f_{pert}(x) & \text{for every } |x| \geq M. \end{cases} \quad (47)$$

The first bound follows simply by compactness (in fact, from the proof of Lemma 10 and the choice (45), we can take $K = \frac{1}{4}e^{-C_q M}$). For the second inequality we need to proceed differently, according to the sign of x . This is due to the fact that the asymptotic behaviours of the function $\lambda_{pert} = I[f_{pert}]$, at $+\infty$ and $-\infty$, are different (Lemma 10).

When $x \leq -M$, we can plainly bound $f_{\bar{\mu}}(x)$ by 1 and then, by the equality (41) from Lemma 10, we have the upper bound (47), provided C_q satisfies (45).

On the other hand, as $x \rightarrow +\infty$ we have that the tail distribution function $f_{\bar{\mu}}$ decays sufficiently fast compared to the other functions: using (37) from Lemma 7 jointly with (42) from Lemma 10, one has

$$\lambda_{pert}(x)f_{\bar{\mu}}(x) = O\left(x^{qk-1}e^{-x^q}\right),$$

while $f_{pert}(x) = e^{-C_q x}$. Thus the second inequality (47) is satisfied as soon as M is sufficiently large. This proves 1).

We now focus on 2). Since f_{μ} is a tail distribution function, it cannot be negative and therefore the inequality (40) is actually stronger:

$$f_{\mu}(x) \geq (f_{\bar{\mu}}(x) - \delta f_{pert}(x))^+ \quad \text{for all } x \in \mathbf{R}.$$

Proceeding as above, and using the same notations, we get

$$f_{\Phi[\mu]}(x) \leq P_k\left((\lambda(x) - \delta \lambda_{pert}(x))^+\right) \quad \text{for every } x \in \mathbf{R}$$

and thus, by (44) in Lemma 11, we have

$$\begin{aligned} f_{\Phi[\mu]}(x) &\leq e^{\delta \lambda_{pert}(x)} f_{\bar{\mu}}(x) \\ &= f_{\bar{\mu}}(x) + \left(e^{\delta \lambda_{pert}(x)} - 1\right) f_{\bar{\mu}}(x) \quad \text{for every } x \in \mathbf{R}. \end{aligned}$$

We need to prove that for M large enough the following condition is satisfied: there exists $K > 0$ such that for every $\delta \in (0, 1)$

$$\left(e^{\delta \lambda_{pert}(x)} - 1\right) f_{\bar{\mu}}(x) \leq \begin{cases} K\delta & \text{for every } |x| < M, \\ \frac{\delta}{2} f_{pert}(x) & \text{for every } |x| \geq M. \end{cases} \quad (48)$$

The first inequality follows again by compactness. Let $x \leq -M$; by the choice (45) of C_q , using (41), we observe that $\delta \lambda_{pert}(x)$ is upper bounded by $\frac{1}{4}$. Then, for such values of x , we have

$$\begin{aligned} \left(e^{\delta \lambda_{pert}(x)} - 1\right) f_{\bar{\mu}}(x) &\leq 2\delta \lambda_{pert}(x) f_{\bar{\mu}}(x) \\ &\leq 2\delta \lambda_{pert}(x) \leq \frac{\delta}{2} f_{pert}(x) \end{aligned}$$

as desired. On the other hand, as $x \rightarrow +\infty$, using (38) from Lemma 7 jointly with (42) from Lemma 10, one has

$$\left(e^{\delta \lambda_{pert}(x)} - 1\right) f_{\bar{\mu}}(x) = o\left(e^{-\frac{1}{2}x^q}\right),$$

while $f_{pert}(x) = e^{-C_q x}$. Thus the second inequality (48) is satisfied as soon as M is sufficiently large. This proves 2). \square

We have concluded the proof of Lemma 8, and thus of Proposition 9.

Concluding remarks

Remark 13. The rather elementary strategy for endogeny that we have shown here seems to require a little “complexity” of the models. For the problem of random metrics (Theorem 2), we require the graph to be non-pivotal; for the mean-field optimization problems (Theorem 3) we require the pseudo-dimension to be greater than 1. If these conditions are not satisfied, the stationary measures for the associated RDEs have exponential tail decays, which do not guarantee the assumption of superexponential decay of the center A7). This last condition is essential in our proof of Lemma 3, with which we control the displacement of the center (Lemma 3). It seems that with the simple exponential tail decay, our method should be completed with a finer asymptotic analysis. The result of [4] suggests the endogeny must hold also in these cases.

Remark 14. It seems very plausible that with the employment of the adapted classes (Definition 12) the arguments of [17] can be carried out to give an alternative proof of Theorem 5.

Acknowledgements

The authors thank Mikhail Khristoforov for the discussions on the cut-off method arisen during their common work [9], and David Aldous for his suggestions. This work has been written during a visit of V.K. to the Universidade Federal Fluminense supported by the Réseau France-Brésil in Mathematics. V.K. thanks also the support of the RFBR project 16-01-00748-a, the hospitality of University of Auckland, and both authors acknowledge the hospitality of CIRM.

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